Four Dimensional Holomorphic Theories

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ABSTRACT

This thesis is devoted to the study of four dimensional theories with a peculiar holomorphic symmetry, analogous to the one of two dimensional conformal theories with current algebras. They are related to the target space theories of $N = 2$ strings. It was recently suggested that the possible higher-dimensional counterparts of conformal theories seem to describe the dynamics of membranes and three-branes. They provide a heuristic explanation of the various string theory dualities and might prove useful in providing a microscopic description of $M$- and $F$- theory together with the conventional strings. We find that the target space theory of $N = 2$ strings seems to be characterized by two principles: holomorphy and anomalies.

Our models have an infinite-dimensional sector of correlation functions and observables in a dynamical and non-trivial theory (unlike the one in topological theories), which is soluble by the finite-dimensional methods. We compute exact effective actions and correlations functions.

There is an interplay between our models and supersymmetric gauge theories in four dimensions. The results of this thesis allow one to extract an exact information about the behaviour of the chiral multiplets in the backgrounds of $N = 1$ vector multiplets. Conversely, one can use the recent results in $N = 2$ super-Yang-Mills to compute the topological quantities, relevant to the holomorphic blocks of our models. In particular, we present a conjectural counterpart of Verlinde formula for $K3$ surfaces.
The organization of the thesis is the following: in the chapter 1 we review a few relevant subjects: two dimensional conformal theories, four dimensional supersymmetric gauge theories, $N = 2$ strings with the emphasis on the features which will be used and generalized. The specialists may skip it and move to the chapter 2 where we introduce the models we are going to study: $WZW$-like theories and free $bc$-systems. The chapter 3 is devoted to the local aspects of the theories with holomorphic symmetry: current algebras, local Hilbert spaces, vertex operators, surface operators, Ward identities and correlation functions in the simple backgrounds. The chapter 4 deals with more involved subject: the global behaviour of the holomorphic theory - the current correlators in the non-trivial background, holomorphic blocks. We list various descriptions of the holomorphic blocks originating in the different representations of the holomorphic bundles. The chapter 5 attacks the problem of counting the number of holomorphic blocks. We introduce the five-dimensional theory, whose partition function is equal to this number and investigate various approaches to its actual computations. As a check, we apply our general methods to the two dimensional problem and re-derive Verlinde formula. In the chapter 6 we make a step towards the complete description of the target space theory of $N = 2$ string and investigate the gravity theories, which are induced by the models we were exploring so far. We uncover the cocycle structures and compute the effective actions replacing in four dimensions the Liouville theory. The chapter 7 is devoted to the integrable models, which accompany holomorphic theories. In the chapter 8 we summarize our results and discuss the future possible directions of development.
# TABLE OF CONTENTS

Abstract iii

Acknowledgements ix

1. Introduction 1
   1.1 Two dimensional Rational Conformal Field Theories 4
   1.2 Four dimensional Supersymmetric Gauge Theories 8
   1.3 $N = 2$ strings 10

2. Holomorphic Symmetry 12
   2.1 Four dimensional avatar of WZW theory 12
   2.2 Free models: bo-systems 15
   2.3 Generalized WZW theories 22

3. Local Theory 26
   3.1 Conserved charges and current algebra 26
   3.2 Construction of the symmetry group 28
   3.3 Simple correlation functions 36
   3.4 Locality, Point-Like and Surface Operators 39
   3.5 Free systems and generalized WZW theories: the local theory 42

4. Global Theory 47
   4.1 Determinant bundles and Quillen anomalies 47
   4.2 Holomorphic blocks 48
   4.3 Non-trivial bundles and Čech approach 52
   4.4 Alternative descriptions of holomorphic blocks 56
   4.5 Projected theories 59

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Disclaimer: In the unpublished version the page numbers may not correspond to their table of content values.
4.6 Hecke operations 61
4.7 Algebraic sector 63
4.8 Appendix. Holomorphic bundles and their moduli 69
4.9 Appendix. Kähler manifolds and twisted supersymmetry 74

5. Topological sector 75

5.1 Instanton theta functions 78
5.2 Three, Five, ... - dimensional Chern-Simons theories 79
5.3 Index counting 91
5.4 Donaldson theory to the rescue 97
5.5 Two dimensional Verlinde formula revisited 104
5.6 Elliptic genera and generalizations 110
5.7 Appendix. Twisted $N = 2$ Theory, Localization, ... 116

6. $N = 2$ strings and holomorphic gravity 125

6.1 Target space theories of $N = 2$ strings 125
6.2 Gravitational sector of a holomorphic theory 126
6.3 Equations of motion from the Chern-Simons theory 129
6.4 Torsion free sheaves 130
6.5 One-loop finiteness of four dimensional avatar of $WZW_2$ 131

7. Integrability and holomorphy 133

7.1 Two dimensional models: Hitchin systems and their degenerations 134
7.2 Four dimensional models: ALE spaces and instantons 159
7.3 Twistor transform 163

8. Further directions and conclusions 165

8.1 Results 165
8.2 Further directions 166

References 167
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1. Introduction

Quantum field theory is a way of describing the fundamental interactions which are responsible for forming the observed matter in a wide range of scales (and is confirmed experimentally up to the energy of $\sim 1\ TeV$). These are the electroweak and strong interactions, which are mediated by the corresponding particles ($\gamma, W^\pm, Z^0, \text{gluons}$). An important property of these interactions is that they are characterised by the dimensionless couplings $\frac{g^2}{4\pi} \sim \frac{1}{12\pi^2} \div \frac{1}{10}$ (the strength of an interaction typically changes with the energies of the processes).

Classically we also see the gravity, which is very relevant macroscopically but fortunately for the quantum field theory is irrelevant as long as the energy scale is small in comparison to the Planck's scale ($M_{Pl} \sim 10^{19}\text{GeV}$). Gravity has a dimensionfull coupling (Newton's constant $G_N = M_{Pl}^2$). In the processes where the characteristic energy involved is $E$ the effective coupling would be $G_N E^2$. Quantum mechanically one expects the virtual particles of arbitrary high energy to be created and this power-like growth of the strength makes gravity ill-defined as a quantum theory.

The problem here is the renormalizability. Although gravity as well as other interactions is a field theory the known Feynmann rules of perturbation theory when applied to gravitational fields lead to the divergencies which become worse as one takes higher and higher orders in coupling.

A similar thing was happening to the Fermi interaction which was rather accurately describing in the 30’s the muon and neutrons decays. Although as a field theory it was non-renormalizable and had a dimensionfull coupling constant ($G_F = 10^{-5}m_p^{-2}$). The theory was fine as long as energies involved were smaller then the Fermi scale ($\sim G_F^{-\frac{2}{3}} \sim 100\text{GeV}$). It turned out that one can save the day if one assumes the existence of an intermediate particle which has a mass of the order of Fermi mass and which serves as carrier of the Fermi interaction. This particle ($W$ boson) was observed experimentally only a couple of decades ago. It took a few decades to build a consistent theory, which had at small energies massive gauge bosons, and which was renormalizable.

It is possible to hope that the extra particles of some sort can cure the problems with gravity. In a sense, this is what happens in string theory. The elementary object in string (perturbation) theory is a one-dimensional string rather then point-like particle. It has a tension (which provides a mass scale) and one can view it as an infinite tower of particles whose masses grow with the oscillator number. One can also work with different kinds of strings: open or closed ones, fermionic and heterotic strings [1], [2].
How one describes the string (perturbation) theory? The general scheme is that one takes a (super)conformal field theory on a worldsheet and couples it to a two-dimensional (super)gravity [3]. Two-dimensional gravity has no degrees of freedom so the coupling to gravity only imposes some constraints on the states in the conformal theory. The correlation functions of the allowed operators give rise to a measure on a (super)moduli space of (super)Riemann surfaces. The $g$-loop Feynmann diagram of the field theory is replaced by the integral of this measure over the (super)moduli space.

It turns out that the equations of motion of a field theory in a string theory context are replaced by the conditions of conformal invariance of the worldsheet field theory. This was, in fact, one of the first indications that (what is now called the conventional) strings, though not useful for describing hadron interactions, can describe gravity, since it turns out that for the two dimensional non-linear sigma-model (whose action has the form $\int d^2 x g_{\mu \nu} \partial_x X^\mu \partial_x X^\nu$) the one-loop $\beta$-function is the Ricci tensor $R_{\mu \nu}$ of the target-space metric. In the case of superstrings there are more restrictive conditions, like the conditions of unbroken supersymmetry which lead to the special solutions of the equations of motion (the relation to the general solutions is similar to the relation of the instantons to the general solutions of the Yang-Mills equations).

This led to the assertion [4] that the two dimensional conformal field theories should be thought of as being the vacua of string theory. Then in order to classify the string vacua one faces the question of classification of conformal field theories which is a hardly soluble problem. It turns out that one can describe a class of conformal theories - the so-called rational conformal field theories (this is not the solution to the problem of classification of all CFT) [5].

The course of development of string theory led to the convergence of various circles of ideas, involving anomalies (together with group and algebraic cocycles), supersymmetry and holomorphy. In the presented thesis this philosophy is supported further. In a sense, we shall unify the subjects and methods of rational conformal theories, supersymmetric gauge theories, non-linear sigma-models in four dimensions and target space theories for $N = 2$ strings via the anomalies and holomorphy.

The theories we shall study will be called holomorphic theories.

In the next few subsections we shall review a few subjects, mainly with to recall some ideas and formulas which will be generalized or used in the main body of the thesis.
1.1. Two Dimensional Rational Conformal Theories

Two-dimensional rational conformal field theories are completely solvable and have formed the basis for much progress in understanding quantum field theory and critical phenomena [6], [7], [8], [9], [10], [5].

We cannot give here the review of the two dimensional conformal theories, so we just mention a few points, which will be important in our higher dimensional generalizations.

The object of vital importance is the chiral algebra. Depending on the conventions it contains the fields of the dimensions $(j,0)$ with $j$ integer (or half-integer). In conformal theory it always contains the Virasoro algebra but can be extended by the fields of various spins. For example, the free $bc$ system of a spin $j$ has the action:

$$S = \int b_j \bar{\partial} c_{1-j}$$

(1.1)

the stress energy tensor

$$T = j : b \partial c : + (1 - j) : \partial bc :$$

(1.2)

which form a Virasoro algebra with the central charge $6j^2 - 6j + 1$. The chiral algebra may be actually bigger. For complex $b, c$ it contains the current algebra $J = : bc :$ at the level 1. For $j$ positive integer it contain the field $b$, and for $j$ negative integer it contains the field $c$. There are also operators of the form $+ \ldots : \partial^m b \partial^{n-m} c : + \ldots$, which form $\mathcal{W}$-algebra. If there are several copies of the $b,c$-system (several flavours) then the chiral algebra contains extra fields of higher spins (giving rise to an extended version of $\mathcal{W}$-algebra). $bc$ systems are the examples of the chiral theories. Usually the theory contains both left and right movers and there are two chiral algebras $\mathcal{A}_L$ and $\mathcal{A}_R$. The simplest example is the theory of a free scalar boson on a circle of the radius $R$. The action is

$$S = \frac{R^2}{4\pi} \int \Sigma \partial \phi \bar{\partial} \phi$$

where $\phi$ is identified with $\phi + 2\pi$. The chiral algebra contains the current algebra at the level $k$, $R^2 = k$, generated by the left current $j_L = k \partial \phi$, and the right current $j_R = k \bar{\partial} \phi$. The left-moving Virasoro algebra is generated by the stress-energy tensor $T = k : (\partial \phi)^2 :$. The central charge of the Virasoro algebra is 1. For the rational $k = p/q$ there is a field of higher spin:

$$F_j = : e^{\bar{\partial} \phi L} : , \quad j = p/q$$

where $k \phi_L(z) = \int^z j_L$. 

11
Less trivial examples are the free fields with the background charge, where the action contains the non-minimal coupling to the background metric, namely:

\[ S = \frac{k}{4\pi} \int_{\Sigma} \partial \phi \overline{\partial} \phi + i \alpha R \phi \]

One can also combine left- and right- \(bc\) system to get non-chiral model.

Another example of major importance is the two dimensional \(WZW\) theory. For any semi-simple group \(G\), and positive integer \(k\) one considers a theory with the action [11]:

\[ S[g] = k \mathcal{L}(g) \]

\[ \mathcal{L}(g) = \frac{1}{8\pi} \int_{\Sigma} \text{Tr} g^{-1} dg \wedge * (g^{-1} dg) + \frac{i}{12\pi} \int_{\Sigma \times I} \omega \wedge \text{Tr} (g^{-1} dg)^3 \]  

(1.3)

where \(g\) is a map from \(\Sigma\) to the group \(G\), \(I\) is the interval \([0, 1]\) and the second term is written for a homotopy \(\tilde{g} : \Sigma \times I \rightarrow G\) such that \(\tilde{g}(x, 0) = g(x), \tilde{g}(x, 1) = 1\). This theory has both left and right chiral algebras. The left algebra is generated by the current algebra:

\[ J \sim kg^{-1} \partial g \quad J^a(z)J^b(w) \sim \frac{k}{(z-w)^2} + \frac{f^{ab}_c J^c(w)}{(z-w)} + \text{reg. terms} \]  

(1.4)

which gives rise to the Virasoro algebra via the Sugawara construction:

\[ T = \frac{1}{k + h^\vee} : \text{Tr} J^2 : \quad T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \text{reg. terms} \]  

(1.5)

with the central charge \(c = \text{dim} \frac{kG}{k + h^\vee}\), where \(h^\vee\) is the dual Coxeter number \((N\) for \(G = SU(N)\)). The right current \(J_R = k \phi^{-1} g^{-1}\).

In the course of study of conformal theory there are two phases: the local theory and the global one. The conformal field theories are studied by means of the Ward identities. The local theory associates a Hilbert space \(\mathcal{H}\) to a point \(P \in \Sigma\) and a small circle \(S\) around it [12], [13], [7]. This space \(\mathcal{H}\) is acted on by the chiral algebras. It can be decomposed as a sum over the irreducible representations:

\[ \mathcal{H} = \sum_{i,j} V_i \otimes \tilde{V}_j \]

where \(V_i\) are irreps of \(A_L\) and \(\tilde{V}_j\) are the ones of \(A_R\) (one of the definitions of the rational theory is the finiteness of the number of irreducibles with respect to the full chiral algebra

12
in this decomposition. Given a field $F_j$ of spin $j$ in the left chiral algebra $\mathcal{A}_L$ and a holomorphic $1-j$-differential $\xi_{1-j}$ one constructs the conserved charge

$$Q_{\xi} = \int_S \xi_{1-j} F_j$$

which by the contour deformation argument is unchanged as the contour $S$ varies, as long as it doesn’t hit other operators or the poles of $\xi$. In particular, if $\xi$ extends holomorphically inside $S$, the charge $Q_{\xi}$ annihilates the state, which corresponds to the path integral over the fields on a disc, containing the point $P$ and bounded by $S$. This state (vacuum) $|0\rangle_L \otimes |0\rangle_R$ is in fact completely determined by the full set of conserved charges in the left- and right- chiral algebras.

The main issue of a global theory is the construction of a correlation function on a general Riemann surface $\Sigma$ [7], [13]. It could be done as follows: cut a point $P$ out of $\Sigma$ and consider a small circle $S$ around it. The path integral over the disc, bounded by $S$ is determined by the conservation laws. The one over the fields outside of the disc is almost determined, up to the finite-dimensional space. This happens because there might be differentials $\xi$ which do not extend holomorphically neither inside $S$ nor outside. This space (which is called a space of conformal blocks) is related to the moduli space of some problem. For example, if the chiral algebra in question is the current algebra this space can be identified with the space of holomorphic sections of a certain line bundle over the moduli space of flat connections with the group $G$ over $\Sigma$. For the spin 2 the moduli space is the moduli space of complex (or projective) structures. For the higher spin chiral algebras the space (which is called a moduli space of $\mathcal{W}$-projective structures) is not well understood. The condition of rationality of the theory implies, in particular, that the number of the blocks is finite.

Now suppose the problem is solved - we get the partition function (or correlation function), more precisely, its chiral part as a vector in some finite-dimensional space. The same is true for the anti-chiral function. The question is how to glue them together. The ambiguity of choosing the vectors in the spaces of conformal blocks is fixed by the fact, that the answer should be well-defined and single-valued functional of the background fields. This imposes some severe restrictions on the way the choice of the vector depends on, say, complex structure of $\Sigma$ and in particular implies the unitarity of Knizhnik-Zamolodchikov-Bernard connection.
1.2. Supersymmetric theories in four dimensions

Another motivation for our study comes from the supersymmetric gauge theories in four dimensions. They also provide a few techniques we are going to use.

Supersymmetry is a theoretic invention of 70's, which was never confirmed experimentally up to now. Yet, it is believed that the most promising candidate of the unified theory of matter and interactions involves supersymmetry.

Supersymmetry relates particles of different statistics. The simplest realization of supersymmetry algebra involves spinor charges $Q_\alpha$ and $Q_\dot{\alpha}$, which commute on the translation generators:

$$\{Q_\alpha, Q_\beta\} = \sigma^\mu_{\alpha\dot{\beta}} P_\mu$$

One can have several copies of this algebra, usually called extended supersymmetry. For example, $N = 1$ susy in four dimensions has 4 generators, 2 left and 2 right, $N = 2$ has 8 and so on.

Lorentz transformations, translations and supersymmetry transformations form super-Poincare algebra (superalgebra). Its irreducible representations contain both bosons and fermions, which are called superpartners and in the case of unbroken supersymmetry all these particles have the same mass.

If we restrict ourselves on the multiplets containing particles of the spin not exceeding 1 then we are bound to have $N \leq 4$ susy. In the simplest case $N = 1$ there are two types of massless multiplets: vector and chiral ones. When one goes to $N = 2$ there are two possibilities. If the multiplet contains particle of the spin 1 it is called a vector $N = 2$ multiplet. It decomposes with respect to $N = 1$ subalgebra into the sum of vector and chiral $N = 1$ multiplets. Another kind of $N = 2$ multiplets has only particles of the spin $\leq \frac{1}{2}$. It consists of two $N = 1$ chiral multiplets and is called $N = 2$ hypermultiplet.

The special property of the supersymmetric theories is the rigidity of the geometry the fields can take values in. Another helpful property is the electric-magnetic duality. Whenever the low energy theory is in the Coulomb phase, i.e. when the gauge group is broken to the abelian subgroup there is a freedom of choosing different kinds of fields as fundamental ones - one can take the original photon or one can dualize and go to the magnetic description where the electric charges are replaced by the magnetic ones. The phenomenon of the duality implies that the geometry of the space of vacua has always the structure of the base of some $Sp(n, \mathbb{Z})$ bundle and various characteristics of the theory are encoded in the properties of this bundle. Together with the holomorphy which follows
from the supersymmetry this enables one to get exact results on the low-energy effective actions of various gauge theories with matter in different representations.

Apart from the obvious physical implications of these exact results the supersymmetric theories provide a powerful tool in studying the topology and geometry of the space-times they are formulated on. In fact, one can formulate a twisted version of the any $N = 2$ supersymmetric theory where at least one supercharge remains unbroken on any four-manifold. This proceeds as follows. The symmetry group of any Lorentz-invariant theory on a four-manifold is $SO(4)$ (we work in Euclidean signature) which is locally isomorphic to $SU(2)_L \times SU(2)_R$. The $N = 2$ susy has also an internal symmetry group $SU(2)_I$ which exchanges the generators of the two $N = 1$ subalgebras. The twisted theory is formulated in such a way that the Lorentz transformations of the fields are accompanied by a rotation inside this $SU(2)_I$ group. In another words, the symmetry group of the twisted theory (on $\mathbb{R}^4$) is the product of the old $SU(2)_L$ and the diagonal subgroup $SU(2)_R \subset SU(2)_R \times SU(2)_I$. This makes the generators of the supersymmetry transformations which used to be left- and right-handed spinors and formed $(2,1,2) \oplus (1,2,2)$ of the $SU(2)_L \times SU(2)_R \times SU(2)_I$ into the $(1,1) \oplus (1,3) \oplus (2,2)$. In other words, the supercharges $Q_\alpha, Q_{\dot{\alpha}}$ become a scalar $Q$, a self-dual two form $Q^\mu_{\rho}$ and a vector $G_\mu$. The scalar survives when the background metric is curved. When the background metric is special, more supercharges survive. In fact, most of the theories we will be studying are naturally formulated on a Kähler manifolds, whose holonomy group (the structure group of a tangent bundle) is reduced to $U(2)$. On a Kähler manifold one has a self-dual two form $\omega$ which is related to metric. In that case one of the components of the self-dual form charge is also unbroken.

1.3. $N = 2$ strings

One of the most intriguing string theories are the so-called $N = 2$ strings, which have $N = 2$ supergravity on the worldsheet (there are heterotic versions with $(0,2), (1,2)$ supergravity, there is also an open $N = 2$ string). Their special property is the cancellation of the string oscillators in all string diagrams, so if one is to make a finite theory of gravity one cannot rely on the "infinite tower of massive particles" as one does in the $N = 1,0$ strings. It means that either $N = 2$ string describes a finite field theory (which is not necessary a renormalizable by the standard arguments) or the Feynmann rules in the theory are different but the content is the same as in the field theory (in other words, the quantity which is calculated needs not be represented as a path integral over a field space).
The target space theory of the $N = 2$ strings was studied in [14] for the closed and heterotic strings. In [14][15] what we will call “the special Kähler point” in the space of WZW-like Lagrangians is related to the classical field theory of the $N = 2$ string. String investigations of this theory have focused on the $S$-matrix for $\pi$ defined by $g = e^{\pi}$. We will study field configurations $g$ related to instantons. Hence one may expect that our results will have a bearing on nonperturbative $N = 2$ string theory.

Recently the heterotic $(2,1)$ strings were proposed as unifying description for the strings and certain types of membranes.

Also, in the slightly different situation (with topological rather then $N = 2$ strings on Calabi-Yau) the Ray-Singer torsions which we shall encounter in the course of our study were recognized [16].

We are going to study holomorphic theories, hoping eventually to learn about rational theories in higher dimensions, supersymmetry and $N = 2$ strings. The main tool of our investigation will be the current algebras, their local and global aspects [17], [18], [19].
2. Holomorphic symmetry

In this chapter we are going to describe a class of higher dimensional generalizations of rational conformal field theories. The first theory will be just a four dimensional analogue of a two dimensional WZW model [20]. Then we describe the analogues of the free fermions and more general bosystems, and then present more general higher dimensional WZW theories. We mostly concentrate on the classical properties of the theories, describe their Lagrangians and establish that they possess a holomorphic symmetry.

2.1. Four dimensional avatar of two dimensional WZW theory

We follow [20].

Lagrangian. We take our space-time $\Sigma_4$ to be a four-dimensional Riemannian manifold equipped with a metric $h_{\mu\nu}$ and a closed 2-form $\omega = \frac{1}{2} \omega_{\mu\nu} dx^\mu \wedge dx^\nu \in \Omega^2(\Sigma_4; \mathbb{R})$. We work in Euclidean signature. The field $g$ in the model take values in some Lie group $G$. In general, for a given $\Sigma_4$ there are several classes of the maps of $\Sigma_4$ to $G$, and in order to write down a Lagrangian in a given class we need to fix a reference field configuration $g_0(x)$. Then, for any $g(x)$ in the same homotopy class as $g_0(x)$ we may define a natural analog of the D=2 WZW theory by the Lagrangian:

$$ S_{\omega}[g; g_0] = \frac{f_5^2}{8\pi} \int_{\Sigma_4} Tr g^{-1} dg \wedge \ast(g^{-1} dg) + \frac{i}{12\pi} \int_{X_5} \omega \wedge \text{Tr}(g^{-1} dg)^3 \quad (2.1) $$

Here $f_5$ is a dimensionful parameter. In the integral over $X_5 = \Sigma_4 \times I$ in (2.1) $\omega$ is independent of the fifth coordinate; moreover, we use a homotopy of $g$ to $g_0$. The action is independent of the choice of homotopy up to a multiple of the periods of $\omega$.

Classical Equations of Motion. Kähler Point. The classical equations of motion, following from (2.1), are:

$$ f_5^2 d \ast g^{-1} dg + i\omega \wedge g^{-1} dg \wedge g^{-1} dg = $$

$$ f_5^2 \partial_\mu h^{\mu\nu} \sqrt{h} g^{-1} \partial_\nu g + i\epsilon^{\alpha\beta\gamma\delta} \omega_{\alpha\beta g^{-1} \partial_\gamma g g^{-1} \partial_\delta g} = 0 \quad (2.2) $$

---

1. Our conventions are the following: Differential forms are considered dimensionless, but $dx^\mu$ carries dimension one. Thus, the metric $g_{\mu\nu}$ is dimensionless, but $f_5^2$, $\omega_{\mu\nu}$ are dimension $-2$, etc.

2. Note that $X_5$ is a cylinder, rather than a cone. This is necessary since $\Sigma_4$ might not be cobordant to zero, and since the periods of $\omega$ might be nontrivial. As was noted in [21], the latter fact caused difficulties in finding a “Mickelsson-type” construction [22] of $\widehat{\text{Map}}(\Sigma, G)$ where $\Sigma$ is a Riemann surface. Using a cylinder and a homotopy construction this problem can be overcome ([23] and section 3.2).
They simplify drastically in the case where \( \Sigma_4 \) is a complex four-manifold with Kähler metric with \( \omega \) the associated Kähler form:

\[
\omega = \frac{i}{2} f^2 \lambda d\lambda \wedge d\bar{\lambda}
\]  \hspace{1cm} (2.3)

We refer to this point in the space of Lagrangians as the "Kähler point."

Using standard properties of the Hodge star we may rewrite the action as:

\[
S_{\omega}[g] = -\frac{i}{4\pi} \int_{\Sigma_4} \omega \wedge \text{Tr}(g^{-1} \partial g \wedge g^{-1} \bar{\partial} g) + \frac{i}{12\pi} \int_{\chi_5} \omega \wedge \text{Tr}(g^{-1} dg)^3
\]  \hspace{1cm} (2.4)

The equations of motion following from (2.4) are:

\[
\omega \wedge \bar{\partial}(g^{-1} \partial g) = 0
\]  \hspace{1cm} (2.5)

These equations are known as the Yang equations (when \( g \) is taken to be hermitian), and are equivalent to the self-dual Yang-Mills equations.

**Remark.** The Lagrangian (2.4) was first written by Donaldson [24]. It was studied by V.Nair and J.Schiff as a natural generalization of the 2D CFT/ 3D CSW correspondence [25][26].

Quantization of \([\omega]\) and algebraicity. As usual, the coefficient of the WZ term is quantized. Two different homotopies of \( g \) to \( g_0 \) define a map \( g : S^1 \times \Sigma \rightarrow G \). Consequently, if the group \( G \) is nonabelian, the measure \( \exp iS \) in the path integral is only well-defined if

\[
\omega \wedge \frac{1}{12\pi} \text{Tr}(g^{-1} dg)^3 \in H^5(S^1 \times \Sigma_4; 2\pi \mathbb{Z})
\]  \hspace{1cm} (2.6)

which forces the cohomology class \([\omega]\) to lie in the lattice:

\[
[\omega] \in H^2(\Sigma_4; \mathbb{Z})
\]  \hspace{1cm} (2.7)

The class \([\omega]\) is the four-dimensional analog of \( k \). Note that although \([\omega]\) is quantized the Lagrangian depends on the representative of the class. Since \( \omega \) is of type \((1,1)\) condition (2.7) implies \([\omega]\) \( \in H^2(\Sigma; \mathbb{Z}) \cap H^{1,1}(\Sigma; \mathbb{R}) \) so the metric is Hodge, and, by the Kodaira embedding theorem, if \( \Sigma_4 \) is compact, it must be algebraic [27]. \(^3\) In two space-time dimensions, all compact complex manifolds are the algebraic curves.

---

\(^3\) Curiously, the supersymmetric \( \sigma \) model can only be coupled to \( N = 1 \) supergravity when the target is a Hodge manifold [28], for similar reasons.
The PW formula and the quantum equations of motion. At the “Kähler point” the classical equations of motion have a local “two-loop group” symmetry \( \mathcal{H}G_\mathbb{C} \times \mathcal{H}G_\mathbb{C} \), where \( \mathcal{H}G_\mathbb{C} = \{ g(z^1, z^2) \in G_\mathbb{C} \} \), which takes \( g \rightarrow g_L(z^i)g(z, \bar{z})g_R(z^i) \). As in two dimensions this is related to the Polyakov-Wiegman (PW) formula [10]:

\[
S_\omega[gh] = S_\omega[g] + S_\omega[h] - \frac{i}{2\pi} \int_{\Sigma_4} \omega \wedge \text{Tr}[g^{-1}\partial g \bar{\partial} h h^{-1}]
\]

(2.8)

Assuming invariance of the path integral measure \( Dg \) with respect to the left action of the group Map(\( \Sigma, G \)) (with \( G \) compact) we can derive the quantum equations of motion:

\[
\bar{\partial}\langle J \rangle = 0 \\
J = \omega \wedge g^{-1}\partial g
\]

(2.9)

This model will be referred to as Four dimensional avatar of WZW_{2}.

2.2. Free models: bc-systems

In two dimensions an important role is played by various bc systems. These are the theories where the fields are a pair of the \( j \) and \( 1-j \) differentials \( b = b_z(dz)^j, c = c_z(dz)^{1-j} \) and the action has the form \( S = \int \bar{b}\bar{c}e \). \( b, c \) can take values in various bundles and \( \bar{\partial} \) can be coupled to various gauge fields. Obviously, if \( g \) is a holomorphic automorphism of the bundle \( \bar{\partial}g = 0 \), then the action is invariant under the transformation \( c \rightarrow gc, b \rightarrow bg^{-1} \).

It is easier to generalize the bc-system of spin \( j = 1 \). In four dimensions one can try to write down the similar action. For example, one can start with a pair of fields:

\( b = b_{ij}dz^i \wedge dz^j \in \Omega^{2,0}, \bar{c} = \bar{c}_iz\bar{z}^i \in \Omega^{0,1} \) and write out an action:

\[
S_0 = \int b\bar{c}\bar{\partial}c
\]

This action has a gauge symmetry \( \bar{c} \rightarrow \bar{c} + \bar{\partial}\psi \). It is important that it commutes with the holomorphic symmetry we wish to study. Indeed, if \( g \) satisfies \( \bar{\partial}g = 0 \) then \( g\bar{c} + \bar{\partial}\psi = g(\bar{c} + \bar{\partial}(g^{-1}\psi)) \). All this extends to the case where \( \bar{\partial} \) is coupled to the external gauge field, such that \( \bar{\partial}B = 0 \) (then \( b \) and \( c \) take their values in the holomorphic bundles). In order to fix this symmetry we introduce a ghost \( \phi \) of the statistics, opposite to that of \( b, \bar{c} \) (i.e. for the fermionic \( b, c \) it will be a bosonic field) and a multiplet of the fields of opposite

* In this context the word "avatar" means the analogue
statistics \((e, \bar{\phi})\). Since what we are doing is the gauge fixing we are going to have a BRST operator \(Q\) which transforms \(\bar{e}\) to \(\bar{\partial}_{\bar{\phi}}\) and \(\bar{\phi}\) to \(e\). The gauged fixed action has a form

\[
S = S_0 + \{Q, \Psi\}
\]

where \(\Psi\) is a gauge fermion. For example, if we choose

\[
\Psi = \bar{\phi} \bar{\partial}_{\bar{\phi}} \bar{e}
\]

we would get the action of the twisted chiral \(N = 1\) multiplet. Indeed, in four dimensions the chiral multiplet consists of the Weyl spinor \(\lambda_\alpha\) and a complex scalar \(\phi\). The action has the form:

\[
\bar{\chi}^\alpha \sigma_{\alpha\beta} D_\mu \chi^\beta + D_\mu \bar{\phi} D_\mu \phi
\]

It has the anomalous \(R\)-symmetry \(U(1)_A\): \(\lambda_\alpha \rightarrow e^{i\theta} \lambda_\alpha, \bar{\lambda}_A \rightarrow e^{-i\theta} \bar{\lambda}_A\). On a Kähler manifold, where the Lorentz group is reduced to \(U(1)_R \times SU(2)_L\) one can use this \(R\)-symmetry [29] and declare the new \(U(1)_R\) to be the diagonal in the product the old one and the \(U(1)_A\). The twist takes \(\lambda_\alpha\) into a \((0, 1)\) form \(\bar{e}\) and \(\bar{\chi}^\alpha\) into a pair of a scalar \(e\) and a \((2, 0)\) form \(b\). The scalars are unchanged. One of the supersymmetry generators become a scalar. In fact, it coincides with the BRST operator we introduced earlier.

This provides an example of a situation where the twisted supercharge becomes actually a BRST operator for some gauge symmetry [30], [31].

More generally, on a Kähler manifold \(\Sigma\) of a complex dimension \(d\) we start with a pair of fields \(b_{p+1}, \bar{e}_p\) which are \((p + 1, 0)\) and \((0, p)\) forms, respectively. We also couple them to the background gauge fields, i.e., \(\bar{e}_p\) becomes a section of a vector bundle \(E\), which we take to be holomorphic\(^4\). The dual field \(b_{p+1}\) takes values in a dual bundle \(E^*\). The trial lagrangian is (in the subsequent formulas the pairing between \(E^*\) and \(E\) is implicitly understood)

\[
S_p^0 = \int \omega^{d-p-1} \wedge b_{p+1} \bar{\partial}_A \bar{e}_p
\]

(2.10)

In the sequel we omit the powers of the Kähler form in the integrals, as they can be easily reinstalled back by counting the degrees. Since the derivative \(\bar{\partial}_A\) squares to zero the action (2.10) has a gauge symmetry

\[
\bar{e}_p \rightarrow \bar{e}_p + \bar{\partial}_A \bar{e}_{p-1}, b_{p+1} \rightarrow b_{p+1} + \partial_A \bar{e}_{p+2}
\]

(2.11)

\(^4\) Which means that the operator \(\bar{\partial}_A\) squares to zero
In order to fix the symmetry we will need a sequence of ghosts and ghosts for ghosts. Indeed, let us fix the symmetry (2.11) by imposing a gauge: $\bar{\delta}^\dagger_A \bar{\phi}^\dagger_p = \partial_A b_{p+1} = 0$. In order to do that properly we need the ghosts: $\tilde{\phi}_{p-1}$ and $\chi_{p+2}$ and a pair of auxiliary multiplets: $(b_{p-1}; \phi_{p-1}), (\bar{\epsilon}_{p+2}; \bar{\chi}_{p+2})$. The BRST transformation $Q^{(0)}$ acts as follows:

\begin{align}
Q^{(0)} \bar{\epsilon}^\dagger_p &= \bar{\delta}^\dagger_A \bar{\phi}^\dagger_p - \partial_A b_{p+1} = b_{p-1} & \quad Q^{(0)} \phi_{p-1} = b_{p-1} \\
Q^{(0)} b_{p+1} &= \partial_A \chi_{p+2} & \quad Q^{(0)} \bar{\chi}_{p+2} = \bar{\epsilon}_{p+2} 
\end{align}

(2.12)

where the subscript always denotes the degree of the form, $f_i$ is a form of type $(i,0)$, $\bar{f}_i$ is of the type $(0,i)$. The corrected action with the gauge fixing term assumes the form:

\begin{align}
S^1_p &= S^0_p + \{Q^{(0)}, \int \phi_{p-1} \bar{\delta}^\dagger_A \bar{\epsilon}^\dagger_p + b_{p+1} \bar{\delta}^\dagger_A \bar{\chi}^\dagger_{p+2} \} = \\
&= \int \Sigma b_{p+1} (\partial_A \bar{\epsilon}^\dagger_p + \bar{\delta}^\dagger_A \bar{\epsilon}^\dagger_{p+2}) + b_{p-1} \bar{\delta}^\dagger_A \bar{\epsilon}^\dagger_p + \phi_{p-1} \bar{\delta}^\dagger_A \bar{\phi}^\dagger_{p-1} + \chi_{p+2} \partial_A \bar{\delta}^\dagger_A \bar{\chi}_{p+2}
\end{align}

(2.13)

Now we have new symmetry (we are temporarily disregarding the bosons $\chi$ and $\phi$):

\begin{align}
\bar{\epsilon}_{p+2} &\rightarrow \bar{\epsilon}_{p+2} + \bar{\delta}^\dagger_A \bar{\epsilon}_{p+3}, b_{p-1} \rightarrow b_{p-1} + \partial_A \phi_{p-2}
\end{align}

(2.14)

Fixing this symmetry produces a new BRST operator $Q^{(1)}$ and a new set of fields:

\begin{align}
Q^{(1)} \bar{\epsilon}^\dagger_{p+2} &= \bar{\delta}^\dagger_A \bar{\phi}^\dagger_{p-1} & \quad Q^{(1)} \phi_{p+3} = b_{p+3} \\
Q^{(1)} b_{p-1} &= \partial_A \chi_{p-2} & \quad Q^{(1)} \bar{\chi}_{p-2} = \bar{\epsilon}_{p-2}
\end{align}

(2.15)

This process continues until the degrees of the forms become either higher than $d$ or lower than $0$. The degree counting is very similar to the spin-orbit coupling in the relativistic hydrogen atom. At the $j$'th step one has the action of the form: for $j$ - odd:

\begin{align}
S^j_p = S^{j-1}_p + \{Q^{(j-1)}, \phi_{p-j} \bar{\delta}^\dagger_A \bar{\epsilon}_{p-j+1} + b_{p+j} \bar{\delta}^\dagger_A \bar{\chi}_{p+j+1} \}
\end{align}

(2.16)

with $Q^{(j-1)}$ acting like that:

\begin{align}
Q^{(j-1)} \begin{pmatrix} \bar{\epsilon}^\dagger_{p-j+1} \\ b_{p+j} \\ \bar{\chi}_{p+j+1} \\ \phi_{p-j} \end{pmatrix} = \begin{pmatrix} \bar{\delta}^\dagger_A \bar{\phi}_{p-j} \\ \partial_A \chi_{p+j+1} \\ \bar{\epsilon}_{p+j+1} \\ b_{p-j} \end{pmatrix}
\end{align}

(2.17)

and for $j$ - even:

\begin{align}
S^j_p = S^{j-1}_p + \{Q^{(j-1)}, \phi_{p+j+1} \bar{\delta}^\dagger_A \bar{\epsilon}_{p+j} + b_{p-j+1} \bar{\delta}^\dagger_A \bar{\chi}_{p-j} \}
\end{align}

(2.18)
\[
Q^{(j-1)} \begin{pmatrix}
\bar{c}_{p+j} \\
\bar{b}_{p-j+1} \\
\bar{\chi}_{p-j} \\
\phi_{p-j+1}
\end{pmatrix} = 
\begin{pmatrix}
\bar{\partial}_{p+j} \\
\partial_{p-j+1} \\
\bar{\partial}_{p-j} \\
\bar{c}_{p-j}
\end{pmatrix}
\]

(2.19)

It produces the gauge fixed action of the form

\[
S = \int \Sigma b(\partial_A + \partial_A^\dagger)\bar{c} + \Phi_+ \partial_A \bar{\Phi}_+ + \Phi_- \partial_A \bar{\Phi}_-
\]

(2.20)

where

\[
\begin{pmatrix}
\bar{c} \\
\bar{b}
\end{pmatrix} = \sum_{j \in \mathbb{Z}} \begin{pmatrix}
\bar{c}_{p+2j} \\
b_{p+1+2j}
\end{pmatrix}
\]

(2.21)

\[
\begin{pmatrix}
\bar{\Phi}_+ \\
\Phi_-
\end{pmatrix} = \sum_{j \in \mathbb{Z}} \begin{pmatrix}
\bar{\chi}_{p+j} + \bar{\chi}_{p+1+j} \\
\chi_{p+j} + \phi_{p+1+j}
\end{pmatrix}
\]

(2.22)

The piece \(b(\partial_A + \partial_A^\dagger)\bar{c}\) of the action (2.20) is nothing but the twisted Weyl fermion (where the standard identification of the spinors with the \((0, p)\) and \((p, 0)\) forms is used \([32]\)). The total BRST charge is \(Q_0 = \sum_{j=1}^d Q^{(j-1)}\). The action (2.20) has a new gauge symmetry, acting on the bosons \(\Phi_\pm\). It leads to a sequence of ghosts for ghosts and new BRST operators \(Q_i\). We are not going to write out the full set of auxiliary fields as it is unnecessary for what follows. One expects a nice algebraic structure, having to do with the action of \(sl_2\) algebra.

In fact, one could address the question of whether the symmetry we have been fixing is anomalous. In the thesis we shall not touch this question as all the formulas we are going to derive will be derived without making real use of the \(Q_{BRST}\) operator. We shall always use for the gauge fixing the operator \(\partial_A^\dagger\) which contains the \((1, 0)\) component of the gauge field \(A\) and we will take the regularization where the anti-chiral system has the \(\partial_A\) operator with the same \(A\).

The partition function in such theory is (up to the zero modes issue) the square root of the partition function of the non-chiral model where one adds complex conjugate fields \(f_i^{(p)}*, \bar{f}_i^{(p)}\). The latter computes the Ray-Singer torsion \([33], [34]\):

\[
\prod_p \left( \det_{\Sigma^{(0, p)} \otimes E_1}^{(p)} [\partial \partial^* + \partial^\dagger \partial] \right) \left( p(-1)^p \right)
\]

(2.23)

22
Formally speaking, up to the $Q_{BRST}$-exact terms the partition function of the non-chiral theory is the product of the following determinants:

$$
\prod_p \left( \text{det}'_{\gamma^{(0)} \otimes E} \tilde{\partial}^i \tilde{\partial} \right)^{(-1)^p} = e^F
$$

(2.24)

where “Reg.” means some regularization, which is needed in order to take care of the infinite-dimensional space of zero modes of $\tilde{\partial}^i \tilde{\partial}$. On the other hand, formally:

$$
\text{Tr} (-)^p e^{-t(\tilde{\partial}^i \tilde{\partial} + \tilde{\partial}^i \tilde{\partial})} = \\
\text{Tr} (-)^p e^{-t\tilde{\partial}^i \tilde{\partial}} (1 + \tilde{\partial}^i \tilde{\partial}) \sum_{n=1}^{\infty} \frac{(-t)^n}{n!} (\tilde{\partial}^i \tilde{\partial})^{n-1} \tilde{\partial}^i \\
= \text{Tr} (-)^{p+1} e^{-t\tilde{\partial}^i \tilde{\partial}} + \text{Tr} (p+1)(-)^p
$$

(2.25)

Therefore, the Ray-Singer torsion is the regularized version of the alternated product of the determinants of the $\tilde{\partial}^i \tilde{\partial}$ operators. The $bc$-system action we wrote in (2.20) also reproduces the regularized alternated product of determinants of $\tilde{\partial}^i \tilde{\partial}$.

Let us see explicitly that they coincide in the four-dimensional case. After adding the fields of opposite chirality we arrive at the action (we will skip the powers of $\omega$ which can be easily reconstructed by the dimensional counting):

$$
S = \int_{\Sigma} \text{Tr} \left[ b \tilde{\partial} A \tilde{c} + \tilde{b} \partial A c + e \tilde{\partial} A \tilde{c} + \tilde{e} \partial A c + \tilde{\phi} \tilde{\partial}^i_A \tilde{\partial} \tilde{\phi} + \tilde{\phi} \tilde{\partial}^i_A \partial \tilde{\phi} \right]
$$

One can add a mass term:

$$
\Delta S = \int_{\Sigma} \text{Tr} \left[ m (\tilde{b} b + \tilde{c} c + \tilde{e} e) + m^2 (\tilde{\phi} \phi + \tilde{\phi} \tilde{\phi}) \right]
$$

Now one can integrate out $e, \tilde{e}, c, \tilde{c}, b, \tilde{b}$ fields, giving rise to the determinant:

$$
\text{det}_{\gamma^{(0)} \otimes E} \frac{1}{m} (\tilde{\partial} \tilde{\partial}^i + \tilde{\partial}^i \tilde{\partial} + m^2)
$$

23
The integral over bosons $\phi, \ldots$ gives rise to the determinant in the numerator:

\[
\frac{1}{\det_{\Omega^{0,0} \otimes E \frac{1}{m^2} [(\bar{\delta}^{\dagger} \delta)^2 + m^4]}
\]

Multiplying this by the

\[
1 = \prod_{p} \left[ \det' \Delta \right]^{(-)^p}
\]

and taking the limit $m \to 0$ we arrive at (2.23).

2.3. Generalized WZW theory

As we will see in the next sections the WZW theories are related to the characteristic classes. In fact, the two dimensional as well as its four dimensional generalization we described earlier are the descendants of the second Chern class $\text{Tr} F \wedge F$. It makes sense to consider the higher classes and the corresponding theories. Like the two dimensional WZW action, they arise as effective actions for the fermions (or $bc$ systems of the previous section) in the background gauge fields. Here we merely introduce the models in any number of dimensions and discuss various forms in which one can write the action of the theories. First of all we introduce a few notations and recall some identities:

\[
A_t = e^{-t\phi} \partial e^{t\phi}, \quad \bar{A}_t = -(\partial e^{t\phi}) e^{-t\phi}
\]

\[
\delta A_t = e^{-t\phi} \partial \left[ \int_{0}^{t} d\tau e^{\tau \phi} \delta \phi e^{-\tau \phi} \right] e^{t\phi}
\]

\[
\partial_t A_t = e^{-t\phi} \partial \phi e^{t\phi}, \quad \partial_t \bar{A}_t = e^{t\phi} \partial \phi e^{-t\phi}
\]

The universal WZW lagrangian can be presented in the form:

\[
S = \int_{\Sigma} \int_{0}^{1} dt \text{Tr}(\phi e^{\bar{A}_t})
\]

where it is understood that one has to expand the exponent and pick out the component of the degree $2d$ of the inhomogeneous form. This form is useful for the generalizations, including gravity. Although the action is written for $2d$ - dimensional field $\phi$ it contains
$2d + 1$ integration, as it is usual for WZ terms. Nevertheless, the equations of motion are local and can be expressed in terms of the field $g = e^\phi$ only:

$$
\delta S = \int_0^1 dt \text{Tr} \left[ \delta \phi e^{iA_t} + \phi \int_0^1 d\tau e^{(1-\tau)A_t} \delta A_t e^{iA_t} \right] =
$$

$$
= \text{Tr} (\delta \phi \int_0^1 dt e^{-t\phi} R_t e^{t\phi})
$$

$$
R_t = e^{\partial A_t} + \int_0^1 d\tau \int_t^1 dt' \partial \partial \left[ e^{\theta_{\phi}} \partial e^{(1-\tau)A_\theta} e^{-\theta_{\phi}} \right] =
$$

$$
e^{\partial A_t} + \int_0^1 d\tau \int_0^1 d\theta \left[ e^{\partial A_\theta} \partial e^{(1-\tau)A_\theta} \right] = (2.28)
$$

$$
e^{\partial A_t} + \int_0^1 d\tau \int_0^1 d\theta \left[ e^{\partial A_\theta} \partial e^{(1-\tau)A_\theta} \right] =
$$

That is, in dimension $2d$ the variation of the action is:

$$
\delta S = \int_0^1 \text{Tr} g^{-1} \left( \partial (\bar{g} g^{-1}) \right)^d = 0 \quad (2.29)
$$

The form (2.27) of the action is not convenient for the studies of the symmetry properties of the theory. Let us discuss another forms one can put the action to. In two dimensional case one has

$$
S = \int_0^1 \text{Tr} \bar{\phi} A_t = \int_0^1 dt e^{tb} \partial \bar{\phi} e^{-t\phi} \int_0^1 d\tau e^{t\phi} \partial \phi e^{-t\phi} =
$$

$$
\int_0^1 \text{Tr} g^{-1} \partial g^{-1} \bar{g} + \int_0^1 dt \text{Tr} (e^{-t\phi} \partial e^{t\phi}) (e^{-t\phi} \partial e^{t\phi})
$$

(2.30)

One can rewrite the last term as

$$
\frac{1}{3} \int_{\Sigma \times I} \text{Tr} (\bar{g}^{-1} d\bar{g})^3
$$

where $\bar{g} = e^{tb}$. Finally, one notices that up to the integer multiples of $2\pi i$ the integral over $\Sigma \times I$ is independent on the choice of $\bar{g}$ as long as the boundary values are the same: $\bar{g}_{t=0} = 1, \bar{g}_{t=1} = e^{\phi}$. This is the usual form of the WZW action in two dimensions.
and when the lagrangian is multiplied by \( \omega \) it becomes the one of the four dimensional generalization of \( WZW_2 \). The next lagrangian has the form:

\[
S_4 = \int_0^1 dt \text{Tr}(\phi \bar{\delta} A_t \bar{\delta} A_t)
\]

Again, we claim that one can rewrite it as a sum of a local terms (the ones which do not contain the integration over \( t \)) and a term \( \frac{1}{5} \text{Tr}(\bar{g}^{-1} \delta g)^5 \) \cite{[35]}.

Indeed, it turns out that \cite{[35]}:

\[
S_4 = \int_\Sigma \text{Tr}(A \bar{A} - \bar{A} A) \bar{\delta} A + \frac{1}{2} (\bar{A} A)^2 + \int_{\Sigma \times I} \frac{1}{5} \text{Tr}(\bar{g}^{-1} \delta g)^5
\]

for \( A = g^{-1} \partial g, \bar{A} = g^{-1} \partial g \), and \( \bar{g} : \Sigma \times I \to G \).

The theory has a holomorphic symmetry \( g \to h_L(z) gh_R(z) \), as it is evident from the manipulations (2.28). Indeed, we have proved there, that

\[
\delta S = \int_\Sigma \text{Tr} \delta g g^{-1} (\partial (\bar{\delta} g g^{-1}))^d = \int_{\Sigma \times I} \text{Tr} g^{-1} \delta g (\bar{\delta} (g^{-1} \partial g))^d
\]

Thus, the Noether currents are: \( J = g^{-1} \partial g (\bar{\delta} g g^{-1})^{d-1} \) and \( \bar{J} = \bar{\delta} g g^{-1} (\partial (\bar{\delta} g g^{-1}))^{d-1} \). One can also define the currents by introducing a background gauge field \( \bar{A} \) and expanding in \( \bar{A} \) the covariant form of the action. On this way one would get:

\[
J = \int_0^1 dt A_t \sum_{m=0}^{d-1} (\partial A_t)^m \phi (\bar{\delta} A_t)^{d-m-1} \quad (2.32)
\]

For \( d = 1 \) both definitions give the same answer. For \( d = 2 \) the covariant definition gives:

\[
J_{d=2} = \int_0^1 dt A_t \phi \bar{\partial} A_t + A_t \bar{\partial} A_t \phi \quad (2.33)
\]

They form the algebra, some version of which has been introduced in \cite{[36]}, \cite{[37]}, \cite{[22]}.
3. Local theory

3.1. Conserved charges and current algebra

Canonical Approach to the Classical WZW Theory and Current Algebra. We consider a four-manifold with space-time splitting $\Sigma = X_3 \times \mathbb{R}$. The phase space of the model can be identified with $\mathcal{P} = \mathcal{P}roj(X_3, G)$, where the momenta are valued in: $I^a(x) \in \Omega^3(X_3, \mathfrak{g})$ and $\mathfrak{g}$ is the Lie algebra of $G$.

Writing the action $S_\omega$ in first order form we extract the symplectic form:

$$\Omega_\omega = \int_{X_3} \text{Tr} \left[ \delta I \wedge g^{-1} \delta g - \left( I + \frac{1}{4\pi} \omega \wedge g^{-1} dg \right) (g^{-1} \delta g)^2 \right] \quad (3.1)$$

from which we obtain the commutation relations of $I^a(x), g(x)$.

$$[I^a(x), I^b(y)]_\omega = f_c^{ab} (I + \frac{1}{4\pi} \omega \wedge g^{-1} dg)^c g^{(3)}(x - y) \quad (3.2)$$

$$[I^a(x), g(y)]_\omega = g(y) T^a g^{(3)}(x - y)$$

From these relations we can obtain the generalization of two-dimensional current algebra. Form the combination

$$J^a(x) = I^a(x) - \frac{1}{4\pi} \omega \wedge (g^{-1} dg)^a(x) \quad (3.3)$$

For a $\mathfrak{g}$-valued function on $X_3$ $e^a(x)$ it gives rise to the charge:

$$Q(\epsilon) = \int_{X_3} \text{Tr}(\epsilon J) \quad (3.4)$$

The charges $Q(\epsilon)$ obey the following commutation relations:

$$\{Q(\epsilon_1), Q(\epsilon_2)\} = Q([\epsilon_1, \epsilon_2]) + \int_{X_3} \omega \wedge \text{Tr}(\epsilon_1 d\epsilon_2) \quad (3.5)$$

We denote this algebra as $\kappa(X_3, \mathfrak{g}, \omega)$.

Remarks.

1. There are several possible generalizations of 2D current algebra. The above algebra is the one relevant to the algebraic sector of our theories, but, for example, using the commutation relations (3.2) it is possible to form a larger algebra as follows. We can

\[5\] We continue to work with Euclidean signature
form the objects like $I_{\xi} = I - \frac{1}{4\pi}(\omega + \xi) \wedge g^{-1}dg$, where $\xi$ stands for any two-form on $X_3$. These form an algebra of charges $Q_{\xi}(f) = \int_{X_3} \text{Tr}(fI_{\xi})$:

\[
\{Q_{\xi_1}(f_1), Q_{\xi_2}(f_2)\} = Q_{\xi_1+\xi_2}([f_1, f_2]) + \\
+ \int_{X_3} \left[ (\omega + \xi_1)\text{Tr}(f_1df_2) - (\omega + \xi_2)\text{Tr}(f_2df_1) \right]
\]  

(3.6)

2. There is an interesting similarity between the algebra $\kappa(X_3, g, \omega)$ and the algebra discovered in [36][37][38][39] for the case when an abelian gauge field strength is equal to a Kähler form $\omega$. This analogy suggests the existence of a free field interpretation of the algebraic sector (see below) of $WZW_4$. Indeed, such an interpretation of the algebraic subsector of the theory exists and is provided by the bosonization in the special representation, see section 3.5.

Current algebra in the generalized $WZW$ theory. It can be shown, that unlike the previous case the algebra of holomorphic currents is extended by the functionals of $\bar{A}$ and the charges $Q(f)$ form the algebra:

\[
[Q(f_1), Q(f_2)] = Q([f_1, f_2]) + \int_M \text{Tr}\partial\bar{A}f_1\partial f_2
\]

We are not going to explore it in this thesis.

3.2. Construction of the symmetry group

Here we follow [23].

In this section we present an explicit construction for the central extension of the group $\text{Map}(X, G)$ where $X$ is a compact manifold and $G$ is a Lie group. If $X$ is a complex curve we obtain a simple construction of the extension by the Picard variety Pic($X$). The construction is easily adapted to the extension of $\text{Aut}(E)$, the gauge group of automorphisms of a nontrivial vector bundle $E$.

3.2.1. Formulation of the problem

Let $X$ be an $n$-dimensional compact manifold, $G$ a Lie group and $\mathcal{G} = \text{Map}(X, G)$ the space of differentiable maps. It is well-known that when $G$ is simple and simply connected the covering group of $\mathcal{G}$ has a universal central extension by

\[
\mathcal{J} = \Omega^1(X; \mathbb{R})/\mathbb{Z}^1(X)
\]

(3.7)
where $\Omega^j(X)$ is the space of all differentiable $j$-forms on $X$, $Z^j(X)$ is the space of closed $j$-forms, and $Z^j_\mathbb{Z}$ is the space of closed forms with integral periods [40]. Correspondingly, the Lie algebra is extended by the space:

$$J = \Omega^1(X)/d\Omega^0(X) \cong Z^{n-1}(X)^\vee$$

(3.8)

by means of the cocycle: 6

$$\langle \alpha(X, Y), \alpha \rangle = \frac{1}{8\pi^2} \int_X \alpha \wedge \text{Tr}(XdY)$$

(3.9)

for $\alpha \in Z^{n-1}(X)$ and $X, Y \in \Omega^0(X; \mathfrak{g})$.

It has been emphasized in [21] that it would be desirable to make the abstract construction of the universal central extension $\hat{\mathcal{G}}$ more explicit. In this note we give such a construction. It is similar to Mickelsson’s approach [22] for the case $X = S^1$ and follows the ideas of section (4.4) of [40]. A different solution to this problem, for the case when $X$ is a Riemann surface, was recently proposed in [41].

A slight generalization of the above problem replaces the group $\text{Map}(X, G)$ by the group $\text{Aut}(E)$ of gauge transformations of a principal $G$-bundle $E$ over $X$. In order to write down the Lie algebra cocycle one fixes a connection $\nabla$ in the adjoint bundle $ad(E)$ and defines: 7

$$c_\nabla(X, Y) = \frac{1}{8\pi^2} \text{Tr}(X \nabla Y)$$

(3.10)

Our construction generalizes to give the universal central extension of $\text{Aut}(E)$.

3.2.2. General construction

Extension of the universal covering of the group $\text{Map}_0(X, G)$. We begin by constructing the extension of the universal covering $\tilde{U\mathcal{G}}$ of the component of the identity $\mathcal{G}_0 = \text{Map}(X, G)_0$ of $\mathcal{G} = \text{Map}(X, G)$. If $B \subset A$ and $D \subset C$ we let $\text{Map}((A, B); (C, D))$ denote the space of smooth maps $f$ of $A$ to $C$, such that $f(B) \subset D$. Introducing $I = [0, 1]$ we define:

$$\mathcal{P\mathcal{G}} \equiv \text{Map}((X \times I, X \times \{1\}); (G, 1))$$

$$\Omega\mathcal{G} \equiv \text{Map}((X \times I, X \times \{0, 1\}); (G, 1))$$

(3.11)

---

6 $\text{Tr}$ is normalized so that, if $\tilde{\mathcal{G}}$ is the simply connected cover, an integral generator of $H^4(B\tilde{\mathcal{G}}; \mathbb{Z})$ is defined by $\frac{1}{8\pi^2} \text{Tr} F^2$.

7 Under a change of connection by an $ad(E)$-valued one form $A$ the cocycle changes by a coboundary: $c_{\nabla + A} - c_{\nabla} = \delta \epsilon_A$, where $\epsilon_A(X) = \frac{1}{8\pi^2} \text{Tr}(X A)$. 
and let $\Omega_0 \mathcal{G} \subset \Omega \mathcal{G}$ be the component of the identity. The construction is summarized by the diagram:

\[
\begin{array}{c}
1 \\
\uparrow \\
\mathcal{J} \\
\uparrow \\
\Omega_0 \mathcal{G} \\
\uparrow \\
1
\end{array}
\quad
\begin{array}{c}
1 \\
\uparrow \\
\mathcal{N} \\
\uparrow \\
\mathcal{P} \mathcal{G} \\
\uparrow \psi \\
1
\end{array}
\quad
\begin{array}{c}
1 \\
\uparrow \\
\mathcal{U} \mathcal{G} \\
\uparrow \\
\mathcal{U} \mathcal{G} \\
\uparrow \\
1
\end{array}
\quad
\begin{array}{c}
1 \\
\uparrow \\
\mathcal{PG} \\
\uparrow \\
\mathcal{PG} \\
\uparrow \\
1
\end{array}
\] (3.12)

In the rightmost column of (3.12) we represent the group $\mathcal{U} \mathcal{G}$ as a quotient. To obtain the middle line we first construct a topologically trivial extension of $\mathcal{P} \mathcal{G}$ by the space of two-forms $\Omega^2(X \times I)$ using the group law:

\[
(g_1, e_1) \cdot (g_2, e_2) = (g_1 g_2, e_1 + e_2 + C(g_1, g_2))
\] (3.13)

where $C$ is a cocycle given by:

\[
C(g_1, g_2) = \frac{1}{8\pi^2} \text{Tr}(g_1^{-1} dg_1 \wedge dg_2 g_2^{-1})
\] . (3.14)

We would like to construct an embedding $\psi$ of $\Omega_0 \mathcal{G}$ as a normal subgroup of $\mathcal{P} \mathcal{G}$ using the 3-form $\omega_3 = \text{Tr}(g^{-1} dg)^3$ on the group $G$. Accordingly, we choose an extension $\bar{h}(x, t, \bar{t})$ of $h \in \Omega_0 \mathcal{G}$ to $X \times I \times \bar{I}$ such that $\bar{h}(x, t, \bar{t} = 1) = 1$ and write:

\[
\psi(h) \equiv \left( h, \frac{1}{24\pi^2} \int_{\bar{t}} \bar{h}^* \omega_3 \right)
\] . (3.15)

The second entry of (3.15) is an element of $\Omega^2(X \times I)$ which depends on $\bar{h}$. Two choices of extension lead to a difference by the closed two-form: $\frac{1}{24\pi^2} \int_{\gamma \times S^1} \bar{h}^* \omega_3$. This two-form has integral periods since for any cycle $\gamma$ in $Z_2(X \times I, X \times \{0, 1\})$ we have the corresponding period:

\[
\frac{1}{24\pi^2} \int_{\gamma \times S^1} \bar{h}^* \omega_3 \in \mathbb{Z}
\] . (3.16)

Thus the difference for two choices of extension is an element of $Z_2^2(X \times I, X \times \{0, 1\})$, the space of closed two-forms vanishing on $X \times \{0, 1\}$ and having integral periods. Hence we
must extend \( \mathcal{PG} \) in (3.12) by the quotient space \( \mathcal{N} \equiv \Omega^2(X \times I)/Z^2(X \times I, X \times \{0, 1\}) \).

Using the Polyakov-Wiegmann formula

\[
(g_1g_2)^*\omega_3 = g_1^*\omega_3 + g_2^*\omega_3 - d(3\text{Tr}(g_1^{-1}dg_1 \wedge dg_2^{-1})) \tag{3.17}
\]

one easily checks that \( \psi \) is a group homomorphism and that the image is a normal subgroup.

The projection in the middle column of (3.12) is defined by restriction of \( g \) to the boundary and gives \( \mathcal{UG} \equiv \mathcal{PG}/\psi(\Omega_0G) \). Correspondingly, we have a map of the centers \( \mathcal{N} \rightarrow \mathcal{J} \) by integration along \( I \).

Central extension for non-simply-connected \( \mathcal{G} \). When \( \mathcal{G} \) is not simply connected we take a composition of the above extension of the universal covering with the extension of the group \( \mathcal{G} \) by its fundamental group:

\[
1 \rightarrow \pi_1(\mathcal{G}) \rightarrow U\mathcal{G} \rightarrow \mathcal{G} \rightarrow 1 \tag{3.18}
\]

to get the universal central extension of \( \mathcal{G} \):

\[
1 \rightarrow \hat{\mathcal{J}} \rightarrow \hat{\mathcal{G}} \rightarrow \mathcal{G} \rightarrow 1
\]

\[
\hat{\mathcal{J}} \equiv \Omega\mathcal{G}/\psi(\Omega_0\mathcal{G}) \tag{3.19}
\]

Here \( \Omega\mathcal{G} \) is the restriction of the extension \( \mathcal{PG} \) of the group \( \mathcal{PG} \) to its subgroup \( \Omega\mathcal{G} \). In order to show that \( \hat{\mathcal{J}} \) is in the center of \( U\mathcal{G} \) one must use (3.17) and the result that the fundamental group of any Lie group is abelian. In general \( \hat{\mathcal{J}} \) is itself an extension

\[
1 \rightarrow \mathcal{J} \rightarrow \hat{\mathcal{J}} \rightarrow \pi_1(\mathcal{G}) \rightarrow 1 \tag{3.20}
\]

If \( \pi_1(\mathcal{G}) \) has no torsion then \( \hat{\mathcal{J}} \cong \pi_1(\mathcal{G}) \oplus \mathcal{J} \) since the projection of an abelian group to \( \mathbb{Z}^n \) splits.

Extension of \( \text{Aut}(E) \). The above construction generalizes to the gauge group of a nontrivial bundle \( E \) by making the following replacements. 8 Let \( \pi : X \times I \rightarrow X \) be a projection. The group \( \mathcal{PG} \) is replaced by the subgroup of \( \text{Aut}(\pi^*(E)) \) of automorphisms which are trivial at \( t = 1 \). \( \Omega\mathcal{G} \) is replaced by the subgroup of automorphisms which are trivial at \( t = 0, 1 \).

Generalizing the extension in (3.13) requires a choice of connection on \( \text{Aut}(E) \) and is defined by the cocycle: \( c_{\nabla}(g_1, g_2) = \frac{1}{8\pi^2} \text{Tr} g_1^{-1} \nabla g_1 \nabla g_2^{-1} g_2 \). Here \( \nabla \) is a connection on

---

8 We continue to assume that \( \text{Aut}(E) \) is connected.
Aut(E), induced by connection $d + A$ on $E$, so $g^{-1} \nabla g \equiv g^{-1} dg + g^{-1} A g - A$, where $\hat{\nabla}$ is a pullback connection. The generalization of the embedding $\psi$ is given by

$$
\psi_{\hat{\nabla}}(h) = \left( h, \frac{1}{24\pi^2} \int_I \left[ \text{Tr}(\hat{h}^{-1} \hat{\nabla} \hat{h})^3 + 3 \text{Tr} F_{\hat{\nabla}}[\hat{h}^{-1} \hat{\nabla} \hat{h} + (\hat{\nabla} \hat{h}) \hat{h}^{-1}] \right] \right). \tag{3.21}
$$

Here we use the 3-form originally discovered in [42] in connection with multi-dimensional solitons (for a recent application see [43]). The extra terms in the second entry in (3.21) are required in order for $\psi_{\hat{\nabla}}$ to define a group homomorphism, or, equivalently, in order to satisfy the PW formula: $\psi_{\hat{\nabla}}(g_1 g_2) = \psi_{\hat{\nabla}}(g_1) + \psi_{\hat{\nabla}}(g_2) + c \nabla(g_1, g_2)$. Again, two choices of extension of $\hat{h}$ lead to an ambiguity in (3.21) by an element of $Z_2^\mathbb{R}(X \times I, X \times \{0, 1\})$ (the periods are related to characteristic numbers of vector bundles constructed from $E$). So, the group Aut($E$) is extended by the same space $\hat{\mathcal{J}}$.

**Extension when Aut($E$) is not connected.** Now suppose that $\pi_0(\text{Aut}(E))$ is not trivial so Aut($E$) = $\Pi \alpha \text{Aut}(E)$. The construction above generalizes by letting $\mathcal{PG} = \Pi \mathcal{PG}_\alpha$ where, for each component $\alpha$ we choose a standard element $g_0^\alpha$ (with $g_0^\alpha = 1$) and let $\mathcal{PG}_\alpha = \{ g \in \text{Aut}(E) : g(x, t = 1) = g_0^\alpha(x) \}$. The group $\Omega_0 \mathcal{G}$ remains unchanged as does the construction.

**Explicit formulae for the central extension.** In order to make the construction more explicit (and to make contact with the formulae of the section 4.4) we must make two choices. First, for each $\gamma \in \pi_1(\mathcal{G})$, we choose a representative $L(\gamma) \in \Omega \mathcal{G}$. Second, we choose a continuation $\Phi(g)$ of $g$, i.e. a map of sets $\mathcal{G} \to \mathcal{PG}$ that inverts the restriction of $\mathcal{PG}$ to $X$. In general neither $L$ nor $\Phi$ is a homomorphism. Indeed, if $G$ is simple then $\Phi$ cannot be a homomorphism. Elements of the centrally extended group $\hat{\mathcal{G}}$ are left-cosets

$$(g, \Phi(g), 0)(1, L(\gamma), \lambda)\psi(\Omega_0 \mathcal{G}) \subset \hat{\mathcal{PG}}$$

where $\lambda \in \hat{\mathcal{J}}$. Elements of the center $\hat{\mathcal{J}} = \hat{\Omega} \mathcal{G}/\psi(\Omega_0 \mathcal{G})$ of the extension (3.19) are also cosets: $(1, L(\gamma), \lambda)\psi(\Omega_0 \mathcal{G}) \subset \hat{\Omega} \mathcal{G}$.

We now give explicit formulae for the multiplication of the cosets. $\hat{\mathcal{J}}$ is itself a central extension (3.20). Thus, as a set it is the space of pairs $(\gamma, \lambda)$, but the multiplication involves a cocycle: $(\gamma_1, \lambda_1)(\gamma_2, \lambda_2) = (\gamma_1 + \gamma_2, \lambda_1 + \lambda_2 + CL(\gamma_1, \gamma_2))$. The cocycle $CL$ is defined by choosing a homotopy $h$ between the loops $^9 L(\gamma_1 \ast \gamma_2)$ and $L(\gamma_1) \ast L(\gamma_2)$ and writing:

$$
CL(\gamma_1, \gamma_2) = \frac{1}{24\pi^2} \int_{I \times \hat{\mathcal{J}}} h^* \omega_3.
$$

---

$^9$ * stands for the path multiplication.
Multiplication of two cosets (3.22) in $\hat{\mathcal{G}}$ leads to the central element $(\gamma(g_1, g_2), \lambda(g_1, g_2))$ given by:

$$
\gamma(g_1, g_2) := [\varphi_{1,2}] \in \pi_1(\mathcal{G})
$$

$$
\lambda(g_1, g_2) = \int_I C(\Phi(g_1), \Phi(g_2)) + \frac{1}{24\pi^2} \int_{I \times I} \tilde{h}^* \omega_3,
$$

(3.24)

The loop $\varphi_{1,2} \in \Omega \mathcal{G}$ is obtained by “glueing together” the paths $\Phi(g_1) \cdot \Phi(g_2)$ and $\Phi(g_1 \cdot g_2)$. More precisely, denoting by $(\cdot)^{\text{inv}}$ the operation of taking the inverse in the semigroup of paths, $\varphi_{1,2} = (\Phi(g_1) \cdot \Phi(g_2)) \ast (\Phi(g_1 \cdot g_2))^{\text{inv}}$. $\tilde{h}$ is a homotopy in $\mathcal{G}$ between the loop $\varphi_{1,2}$ and the representative of its homotopy class $L([\varphi_{1,2}])$.

**Relation to the descent procedure.** In light of the above results it is instructive to reconsider the descent procedure for constructing gauge group cocycles with values in functionals of gauge connections [36]. Recall that one introduces three operations $d, d^{-1}, \delta$, on differential-form valued functionals of group elements and gauge connections such that

$$
\delta^2 = d^2 = 0, \quad \delta \text{ is a group cochain differential [36]; } d^{-1} \text{ is defined in [44],[19] and is essentially the operation } \int_I \text{ used in equation (3.15) above. Starting from a } 2m\text{-form } \Omega_{2m} = \text{Tr} F^{2m}, \text{ which satisfies } \delta \Omega_{2m} = d \Omega_{2m} = 0, \text{ one applies the operation } d^{-1} \delta \text{ a total of } k \text{ times to get } \\
\Xi = \int_X d^{-1} \delta \cdots d^{-1} \delta d^{-1} \Omega_{2m}, \text{ which satisfies } \delta \Xi = 0 \text{ and hence is a cocycle of degree } k \text{ in dimension } 2m - k - 1.
$$

Comparing with (3.9) it is apparent that one might have started with an $n + 3$-form $\Omega'_{n+3} = \alpha \wedge \text{Tr} F^2$ where $\alpha$ is a closed $n - 1$-form with integer periods. Repeating the descent procedure with the appropriate definition of $d^{-1}$, one obtains a 2-cocycle taking values in $\mathcal{J}$. For example, if $H^3(G)$ is trivial, the form $\text{Tr}(g^{-1} dg)^3$ on the group $G$ is exact: $\text{Tr}(g^{-1} dg)^3 = db_2$, where $b_2$ is a 2-form on a group $G$. The cocycle $C_d$ obtained using the descent procedure is:

$$
C_d(g_1, g_2) = \lambda(g_1, g_2) + \int_I (\Phi(g_1)^* b_2 + \Phi(g_2)^* b_2 - \Phi(g_1 g_2)^* b_2),
$$

(3.25)

and differs from the one presented in this paper by a coboundary. One can show that the cocycle $C_d$ is independent of the choice of the section $\Phi$, while under a change of $b_2$ it changes by a coboundary. We will return to the relation to the descend procedure in the chapter 5 when we will be discussing another sort of the descend appearing in the topological theories.

33
3.2.3. Specializations

Let us now assume that $\pi_1(\mathcal{G})$ has no torsion. We have constructed the universal central extension of $\mathcal{G}$ by $\mathcal{J} \oplus \pi_1(\mathcal{G})$. All other central extensions are formed by taking quotients $\widehat{\mathcal{G}} / \widehat{\mathcal{J}}_1$, where $\widehat{\mathcal{J}}_1 \subset \mathcal{J} \oplus \pi_1(\mathcal{G})$. Here we list some interesting examples.

First, to get a one-dimensional central extension we note that the group of characters of $\mathcal{J} \oplus \pi_1(\mathcal{G})$ is $\mathbb{Z}_2^{-1}(X) \oplus H^1(\mathcal{G})$. Given an element $(\alpha, \chi)$ of this space we can take $\widehat{\mathcal{J}}_1$ to be $\ker(\alpha \oplus \chi)$. The corresponding group extension is given by $\exp[2\pi i \int_X \alpha \wedge \chi(g_1, g_2) + 2\pi i \chi(g_1, g_2)]$. Second, if $X$ is equipped with a metric then we can take $\widehat{\mathcal{J}}_1 = d^* \Omega^2 + Z^1$. In this case the connected component of the center is the torus $H^1(X; \mathbb{R}) / H^1(X; \mathbb{Z})$. Third, if $X$ is also equipped with a complex structure then we can work over the complex numbers and choose $\widehat{\mathcal{J}}_1 = [\partial^! \Omega^{0,2} \oplus \Omega^{1,0}] + Z^2$. In this case the connected component of the center is the connected component of the Picard variety

$$\text{Pic}_0(X) \equiv H^1(X; \mathcal{O}) / H^1(X; \mathbb{Z})$$ \hspace{1cm} (3.26)

Finally, if $X$ is a complex curve we do not need a metric to construct the central extension by the Jacobian of the curve. In this way we obtain a solution of the problem posed in [21] and solved independently using a very different and beautiful technique of holomorphic geometry in [41]. Moreover, since in this case $\pi_1(\mathcal{G}) = \mathbb{Z}$, the center is the full Picard variety.

3.3. Simple correlation functions

3.3.1. Ward identities

The identity (2.8) implies that the $(2,1)$-form current $J = \omega \wedge g^{-1} \partial g$ satisfies:

$$\frac{i}{2\pi} \partial_x \left( J^A(x) \prod_i J^{A_i}(x_i) \right) = \sum_i (\omega \wedge \partial) i (\delta(x, x_i)) \delta^{A_i A_i} \left( \prod_{j \neq i} J^{A_j}(x_j) \right)$$

$$+ \sum_i f^{A_i A_j B_i}(\delta(x, x_i)) \left( J^{B_i}(x_i) \prod_{j \neq i} J^{A_j}(x_j) \right)$$ \hspace{1cm} (3.27)

where $\delta(x, y)$ is a 4-form in $x$ and a 0-form in $y$.

In strong contrast to the situation in 2D, the identities (3.27) do not determine the correlation functions since $\bar{\partial}$ has an $\infty$-dimensional kernel and these identities cannot fully determine the correlation functions. However, the identities, together with simple analyticity arguments do determine many correlation functions in a way analogous to the
2D case. Indeed, suppose \(Y\) is a compact three-manifold, which divides \(X_4\) in two parts: \(X_+\) and \(X_-\), and suppose that \(f\) is a function on \(Y\) taking values in the Lie algebra \(\mathfrak{g}\) and extending holomorphically to \(X_+\). Then the integral

\[
V(f, Y) = \int_Y \text{Tr}(fJ),
\]

(3.28)

generates an infinitesimal gauge transformation. The field \(g\) in the region \(X_+\) gets transformed to \(g + fg\), and remains unchanged in \(X_-\). Now, keeping in mind the fact, that \(J\) transforms as

\[
J \rightarrow J + \omega \wedge \partial f + [J, f],
\]

(3.29)

we can derive the correlation functions of the currents, integrated with appropriate functions \(f\).

One can also derive a Ward identity for the generating function of the current correlators. We will discuss this function in details in the section "Global theory". Here we just extract a simple consequence of PW identity:

\[
\langle \exp(\int_\Sigma \text{Tr}J\bar{\partial}hh^{-1}) \rangle = \exp(-S_{ZW4}(h))
\]

(3.30)

Expanding the right hand side in \(h\) one can obtain an infinite number of correlation functions of the kind we discussed. The all are the correlators of the currents, integrated with appropriate weights over some surfaces.

In the generalized \(WZW\) theories there also are the correlation functions which are determined from the algebraic properties of the currents. As those are more complex then in the "avatar" theories we will study them elsewhere. We mention a few points which make the generalized theory (and the \(bc\) systems with the corresponding anomaly) look more difficult then the "avatar"-like theories.

We shall only treat the abelian case as the relevant complications arise there. Recall that the action has the form:

\[
S(\phi) = \int_\Sigma \partial\phi \bar{\partial}\phi \partial\phi
\]

We want to derive an analogue of the PW formula. We have:

\[
S(\phi_1 + \phi_2) = S(\phi_1) + S(\phi_2) + \int_\Sigma j_1 \bar{J}_2 + J_1 j_2
\]

(3.31)
where we have introduced a "small current" \( j = \partial \phi \) (it is \((1, 0)\) form) and its conjugate \( \bar{j} = \bar{\partial} \phi \). We see, that in the generating function it is natural to consider both \( J \) and \( j \). We define formally:

\[
\mathcal{Z}(\bar{A}, \bar{a}) = \langle \exp(S + \int_{S_i} j \bar{a} + J \bar{A}) \rangle
\]

with \( \bar{A} \) being \((0, 1)\) form and \( \bar{a} \) being \((1, 2)\) form. We have two identities \( \mathcal{Z} \) satisfy:

\[
\left[ \bar{\partial} \frac{\delta}{\delta \bar{A}} - \partial \bar{A} \bar{\partial} \frac{\delta}{\delta \bar{a}} \right] \mathcal{Z} = \partial \bar{a} \mathcal{Z}
\]

and

\[
\left[ \frac{\delta}{\delta \bar{a}} - \frac{\delta}{\delta \bar{a}} \bar{\partial} \frac{\delta}{\delta \bar{a}} \right] \mathcal{Z} = 0
\]

Again, the second equation is non-linear and hardly soluble so we restrict \( \mathcal{Z} \) onto submanifold:

\[
\bar{a} = \bar{A} \partial \bar{A}
\]

and there the functional \( \mathcal{Z} \) is completely determined by the first equations (again, up to the global subtleties). This is the analogue of the definition of the algebraic sector.

### 3.4. Locality, Point-like and Surface Operators

Here we will discuss the issue of locality in our field theories. Certainly, the symmetry which governs our theories is not the gauge symmetry, but viewed as a global symmetry it is infinite-dimensional and is almost as good as a local one. It makes us wonder whether we can define some local objects, whose correlation functions can be, in principle, computed exactly, just from the knowledge of the current correlation functions.

It is clear that the issue of locality has to do with the issue of how one can describe the points or submanifolds of the manifold using only the holomorphic data. The simplest thing is to take a meromorphic function \( f \) and to consider its divisor. It is a union of hypersurfaces with multiplicities where \( f \) has a pole or a zero. If the function takes values in a Lie algebra \( \mathfrak{g} \) then it is natural to consider an operator, associated to the polar divisor \( P_f \). One takes a tubular neighborhood \( T_f \) of \( P_f \) and integrates the current along the boundary of \( T_f \) with the weight \( f \):

\[
\mathcal{J}_f = \int_{\partial T_f} \text{Tr}(fJ)
\]
The integral doesn’t change as one moves the contour $\partial T_f$ as long as it doesn’t cross the divisor $P_f$ (and doesn’t hit another operator), since inside the path integral one can use the Ward identity which states that $\bar{\partial} J = 0$.

Remarks on States and Hilbert Spaces: An operator formalism in 4d

One goal of this investigation is the generalization of concepts of 2D CFT to 4D. In the 2D case one fruitful approach to defining states and Hilbert spaces proceeds by considering states defined by path integrals on 2-folds with boundary, the boundary is interpreted as a spatial slice.

In 2D a compact connected spatial slice is necessarily a copy of $S^1$. In 4D, on the other hand, there is a wide variety of possible notions of space. Two natural choices are $S^3$, corresponding to radial propagation around a point and a circle bundle over an embedded surface, corresponding to radial propagation in the normal bundle to the surface.

Suppose now that the 3-fold bounds a 4-fold $Y = \partial X$, then a particular state $\Psi_Y \in \mathcal{H}(Y)$ is defined by the path integral. An important problem is to describe how one can construct this state explicitly. One immediately encounters a fundamental difficulty in defining the state via conserved charges called the “2/3 problem.”

Conserved charges in 2D come from (anti-) holomorphic currents. If $f|_{X_1}$ extends to a holomorphic function on $X_2$ then by contour deformation:

$$\bar{\partial}(g^{-1} \partial g) = 0 \quad \Rightarrow \quad \oint_{X_1} \text{Tr}(fg^{-1} \partial g) \Psi = 0 = \oint_{X_1} \text{Tr}(f \partial gg^{-1}) \Psi$$  \hspace{1cm} (3.32)

In 2D, “enough” boundary values $f$ extend to holomorphic or anti-holomorphic functions on $X_2$ that the conserved charges determine the state up to a finite-dimensional vector space (related to conformal blocks).

In 4D, $\omega \wedge g^{-1} \partial g$ is a $\bar{\partial}$-closed $(2, 1)$ form, and hence, if $f|_{X_3}$ extends to a holomorphic function on $\Sigma_4$ “contour deformation” of $X_3$ implies

$$\oint_{X_3} \text{Tr} \left[ f(z^1, z^2) \omega \wedge g^{-1} \partial g \right] \Psi = 0$$  \hspace{1cm} (3.33)

but note there are not nearly enough such functions to determine the vacuum. Even in the best cases 10 we are missing one functional degree of freedom: boundary values of holomorphic functions depend on two functional degrees of freedom but to determine the state we need conserved charges depending on three functional degrees of freedom.

One possible approach to the problem uses the twistor transform of the SDYM equations. In the abelian case it does solve the problem of missing charges - harmonic functions on a ball with the flat metric are linear combinations of holomorphic functions in different complex structures.

10 for example, in radial quantization around a point
3.4.1. **Point-like operators**

These operators correspond to the insertions to the points in $\Sigma$. In all types of $WZW$ theories they lead to the modifications of the conservation laws. Indeed, if the Hilbert space $H_P$ is associated to a small sphere $S$ around the point $P$, then the integral of the current $J$ against some function $f$ which extends holomorphically inside $S$ is not zero. Rather, as the interpretation of the current as the generator of the symmetry transformation suggests the result of such integration equals to the transformation of the operator inserted. The natural operators to insert are the matrix elements of the field $g$ in some representation $R$ of the group $G$. As we discussed before, we don’t have enough charges to determine the state one gets by inserting this operator.

3.4.2. **Surface operators**

As the abelian theory suggests, the ”chiral” surface operators could be defined as follows. Consider a codimension two submanifold $D$ and a codimension one submanifold $\Gamma$, such that one of the boundary components of $\Gamma$ is $D$. Then one can define an ”ordered” exponential of the integral of the holomorphic current $J$ along $\Gamma$. In the abelian case one doesn’t care about ordering. For the four dimensional free scalar theory one gets:

\[ V_1(D) \sim \exp \imath \int_{\Gamma} \omega \wedge \partial \phi \]

whereas for the generalized $WZW$ for the group $G = U(1)$:

\[ V_1(D) \sim \exp \imath \int_{\Gamma} \partial \phi \partial \bar{\phi} \]

We have seen, that the generalized theories require both $\bar{A}$ and $\bar{a}$ sources. We have also seen that the soluble sector is distinguished by some relation between $\bar{a}$ and $\bar{A}$. It can be translated to the statement, that the surface operator $V(D)$ must be accompanied by the point-like operator inserted at the self-intersection points $D \cdot D$. All this can be easily deduced from the property (3.31) of the action.

3.5. **Free systems and generalized $WZW$ theories: local theory**

Up to now we have studied the consequences of the peculiar current algebra, which is present in the theory (2.4). Actually, all the arguments concerning Ward identities, current correlation functions, effective action and local operators would apply in any theory, which possess the same algebra.
Here we shall show that bc-system in the special representation has this property. Then we shall move on and consider the current algebra and effective action for the general representation. It turns out that the generalized WZW theory shows up in the formulas for the effective actions. It captures all the relevant properties of the symmetry of the model and allows one to reconstruct the correlation functions essentially from a classical computation.

3.5.1. Evaluation of effective action for bc system

Consider the partition function of the non-chiral bc-system coupled to the holomorphic bundle $E$. It is a functional of the external gauge field - a connection in the bundle $E$. We shall derive an identity, which allows to reconstruct the partition function and study its properties. Let $F$ denotes the free energy. Let $\phi$ be the field in the adjoint representation. We think of $\phi$ as being locally the anti-hermitian matrix-valued function on $\Sigma$. The transformation with $g = e^{i\phi}$ is the unitary gauge transformation and therefore $F$ is preserved. Thus:

$$\partial_A \phi \frac{\delta}{\delta A} F - \bar{\partial}_A \phi \frac{\delta}{\delta A} F = 0$$

On the other hand, $h = e^{i\varphi}$ is a hermitian-valued field and the transformation:

$$A \rightarrow h^{-1} Ah + h^{-1} \partial h \quad \bar{A} \rightarrow h\bar{A}h^{-1} - \bar{\partial}hh^{-1} \quad (3.34)$$

needs not to preserve $F$. We shall compute the result of this transformation using two techniques: index theorem and cocycle considerations.

In the first approach we write $H_\tau = \{ \bar{\partial}, e^{i\tau\phi} \bar{\partial}^i e^{-i\tau\phi} \}$ and:

$$F_\tau = \int_0^\infty \frac{dt}{t} \text{Tr}'(e^{-tH_\tau} (-1)^p p)$$

$$\partial_\tau F = i \int_0^\infty \frac{dt}{t} \text{Tr}'(\bar{\partial}_A \partial_A \phi e^{-tH_\tau} (-1)^p p) =$$

$$= i \int_0^\infty \frac{dt}{dt} \text{Tr}'(\phi e^{-tH_\tau} (-1)^p) =$$

$$= i \int_\Sigma \text{Tr}(\phi e^{-\frac{F}{2\pi t}}) Td(\Sigma) \quad (3.35)$$

where in the last line we used the integrand in the index theorem, as in the heat kernel proof of the index theorem what one actually proves [45], [34] is that the $t^0$ term in the short time expansion of the supersymmetric heat kernel $(\rightarrow)^F e^{-tH}$ has the form we used in (3.35).

39
We see that the generalized WZW action emerges in the leading in the gauge field order. Indeed, let us invert (3.35). Denote by
\[ g^* F(\bar{A}, A) = F(g^{-1} \bar{A} g + g^{-1} \bar{\partial} g, g^i \bar{A} g^{i-1} - \bar{\partial} g^i g^{i-1}). \]
Then for \( gg^\dagger = 1 \) \( g^* F = F \), therefore for any \( g \):
\[ F(g^{-1} \bar{A} g + g^{-1} \bar{\partial} g, g^{-1} A g + g^{-1} \partial g) = F \]
and also
\[ \frac{d}{dt} (e^{\bar{A} \phi})^* F = (e^{\bar{A} \phi})^* \int_{\Sigma} \text{Tr} \phi e^{-\frac{F}{2 \pi i}} T d(\Sigma) \]
\[ \frac{d}{dt} (e^{\bar{A} \phi})^* F = \int_{0}^{1} dt (e^{\bar{A} \phi})^* \int_{\Sigma} \text{Tr} \phi e^{-\frac{F}{2 \pi i}} T d(\Sigma). \]  
(3.36)

Finally:
\[ F(g^{-1} \bar{A} g + g^{-1} \bar{\partial} g, A) = F(\bar{A}, A) = \]
\[ = \int_{0}^{1} dt \int_{\Sigma} \text{Tr} (\phi \exp \frac{[e^{-i \bar{A} \phi} \partial \bar{A} e^{i \bar{A} \phi}, \partial A]}{2\pi i}) T d(\Sigma). \]  
(3.37)

The alternative way of deriving the same result (which is a bit more painful, although conceptually clearer) is the following. Consider a chiral system. Its partition function is naively holomorphic in \( \bar{A} \) (since \( A \) enters only through the \( Q \)-exact terms). On the other hand, the theory has a gauge anomaly, which is given by the cocycle \( \alpha(g, \bar{A}) \) [36], [37], i.e.
\[ Z_{ch}(\bar{A}^g) = e^{i \alpha(g, \bar{A})} Z_{ch}(\bar{A}) \]

Here \( g \) is a unitary gauge transformation but the holomorphy implies that the same equation holds for complex \( g \), too. Now, in order to get a gauge invariant non-chiral theory, one combines the chiral and anti-chiral partition functions, times the possible local counterterms, which allow to get the gauge invariance with respect to the unitary gauge transformations. It means, that there exists a local functional \( \beta(A, \bar{A}) \), such that
\[ \alpha(g, \bar{A}) + \alpha(g^{-1}, A) + \delta g \beta = 0 \]
for unitary \( g \). Here \( \delta g \) is a group coboundary operator:
\[ \delta g \beta(A, \bar{A}) = \beta(A g^i, \bar{A} g^i) - \beta(A, \bar{A}) \]

In two dimensions \( \beta = \int_{\Sigma} \text{Tr} \mathbf{A} \mathbf{A} \). In four dimensions:
\[ \beta = \int_{\Sigma} \text{Tr} \mathbf{A} \bar{\partial} \mathbf{A} + \bar{\mathbf{A}} \mathbf{A} \bar{\partial} \mathbf{A} - \frac{1}{2} (\mathbf{A} \mathbf{A})^2 \]

Finally notice that for the gauge field \( A, \bar{A} \), such that \( F^{0,2} = F^{2,0} = 0 \) the knowledge of the way the answer transforms under the complex gauge transformation implies the dependence of all the local degrees of freedom and the only problem left is the one of the global theory.

40
3.5.2. $bc$ system - avatar of two dimensional chiral fermions

Now we are able to calculate the effective action for the $bc$ system in the arbitrary representation of the gauge group. We want to find the $bc$ system whose effective action would reproduce the action (2.4).

As we noticed before, the effective action arises from the computation of the anomaly, which in turn is nothing but the $\text{ch}_{d+1}$ term in the formal Chern character of the bundle $E$ the $bc$ system is coupled to ($d$ is the complex dimension of $\Sigma$). We claim that if $E = L \otimes (V \oplus V^*)$, for $L$ - a line bundle and $V$ - general holomorphic bundle, then in the dimension $4$ (i.e. $d = 2$) one gets the action of the four dimensional avatar of two dimensional WZW with the Kähler form $\omega$ replaced by the curvature of $L$ plus the abelian generalized WZW action. Indeed,

$$\text{Ch}(E) = e^{\omega_1(L)}(\text{Ch}(V) + \text{Ch}(V^*)) = e^{\omega} \text{Tr}_V(e^{\frac{F}{2\pi}} + e^{-\frac{F}{2\pi}})$$

and therefore $\text{ch}_6(E) = \frac{1}{3} c_1^3(L) - \frac{1}{4\pi^2} c_1(L) \text{Tr}_V F^2$.

Notice, that Donaldson in [46] proposed a virtual bundle $E = V \otimes (L - L^{-1})^{d-1}$ which gives for non-abelian part the action (2.4) in any number of dimensions (with $\omega$ replaced by $\omega^{d-1}$). Physically, the minus sign corresponds to the flip in the statistics and gives rise to a rather exotic systems. Our proposal allows to have the usual fields in 4 dimensions (and the field content is actually the one of $N = 2$ hypermultiplet, with $N = 2$ susy broken down to $N = 1$ by the coupling to $L$).
4. Global theory

4.1. Determinant bundles and Quillen anomalies

In this section we assume that the moduli space of bundles exists and is smooth (we presented a few subtleties in the Appendix).

Over the moduli space of holomorphic bundles there is a natural line bundle $L$ [46], [34]. Its fiber over a point $m$ which is an isomorphism class of the bundles is the one-dimensional space

$$L_m = \otimes_i (\det H^i(S, E_m))^{(-1)^i}$$

The dimensions of the spaces $H^i$ can jump as one varies the point $m$, but the ratio of the top exterior powers of these spaces (which we denoted as $\det$) varies holomorphically with $m$. One can compute the first Chern class of the bundle $L$ in terms of the universal classes of $M$, which are defined as follows.

Consider the space $M \times \Sigma$ and consider a holomorphic bundle $\mathcal{E}$ over it, whose restriction on $\{m\} \times \Sigma$ is the bundle $E_m$. The characteristic classes of $\mathcal{E}$ are the elements of the cohomology group of $M \times \Sigma$ and can be pushed forward to $M$ by integrating along the cycles in $\Sigma$. Thus, each class $c_k(\mathcal{E})$ gives rise to a several classes in $H^*(M)$.

4.1.1. One-loop computations

We are going to compute the variation of some ratio of determinants of Laplace operators, acting in the spaces of $(0, p)$ forms. Actually, we are going to take the free energy, defined as

$$F = \sum_p p(-1)^p \log(\det'(\xi_{0,p})\Delta)$$

and hit it with the equivariant derivatives $\delta'$ and $\delta''$. Recall that their anticommutator is a gauge transformation and since the determinants are gauge invariant the non-trivial answer one gets when acting by commutator of $\delta', \delta''$. The answer will contain $\phi$ as well as $\psi$ and $\bar{\psi}$. Notice that if we know the part containing $\phi$ we can reconstruct the rest by supersymmetry. Therefore, let us concentrate on the contribution of $\phi$. The $\phi$-dependent part of the commutator $[\delta', \delta'']$ is nothing but the axial gauge transformation with the parameter $\phi$ (here by the axial transformation we mean the transformation which makes $\bar{\delta}_A \to \bar{\delta}_A + \bar{\delta}_A \phi$, $\partial_A \to \partial_A - \partial_A \phi$. It shifts the Laplacian by $\bar{\delta}_A^\dagger \bar{\delta}_A \phi$.)

42
Now the change in $F$ can be computed as follows (similarly to what we did in (3.35)):

\[
F = \int_0^\infty dt \frac{t}{t} \text{Tr}(e^{-tH} (-) F F) \]
\[
\delta F = \int_0^\infty dt \frac{1}{t} \text{Tr}(\delta A \phi e^{-tH} (-) F) = \\
\int_0^\infty dt \frac{d}{dt} \text{Tr}(\phi e^{-tH} (-) F) = \\
\int \text{Tr}(\phi e^{-\frac{F}{2\chi}}) T d(\Sigma) \tag{4.1}
\]

This result implies that the leading gauge field contribution to the curvature of the determinant line bundle is represented by the $2d$-observable, constructed out of the $\text{Tr}j^{d+1}$. We shall use this in the next sections, where we shall consider the quantizations of the moduli spaces, and the determinant line bundle shall play the rôle of the prequantization bundle.

### 4.2. Holomorphic blocks

Before discussing the spaces of holomorphic blocks let us digress for a moment and discuss Ward identities in a general field theory. We denote the space-time by $\Sigma$ and a space-like surface by $X$. Given that a theory possesses a (non-anomalous) symmetry group $G$ one is to define a generating function of the correlators of the currents, associated to this symmetry:

\[
\mathcal{Z}(B) = \langle \exp(- \int \Sigma \text{Tr}(BJ)) \rangle \tag{4.2}
\]

(we recall that the current $J$ is always a $d-1$ form for $d$-dimensional $\Sigma$). For any symmetry group $G$ one can derive a functional equation on (4.2), given the commutation relations between the components of the current $J$:

\[
\delta_\epsilon S = \int \Sigma \text{Tr}(d\epsilon J) \\
\left[ d^a B^a + B^c \delta e^a \frac{\delta}{\delta B^c} \right] Z = 0 \tag{4.3}
\]

where $a, c$ are the indices of the Lie algebra of $G$, $\delta_\epsilon$ denotes the infinitesimal local transformation. Suppose, that there are local commutation relations:

\[
\delta_{\epsilon(x)}J^b(y) = (f^c_{ab} \delta^d(x, y) J^c(y) + K^{ab}(x, y))e^a(x) \tag{4.4}
\]
where the distribution $K^{ab}(x, y)$ is supported on the diagonal $x = y$ (it is referred to as Schwinger term). (It must satisfy certain consistency conditions - it should be a cocycle for the Lie algebra of $G$ - more precisely, the charges form an extension of the algebra $G$ (generally, by the functionals of the fields):

$$Q(f) = \int_X \text{Tr}(fJ)$$

$$[Q(f_1), Q(f_2)] = Q([f_1, f_2]) + K(f_1, f_2) \tag{4.5}$$

The condition is that $Q(f)$ and $K(f, g)$ must form a closed algebra. The simplest case is when $K$ is a $c$-number, it corresponds to the central extension of the Lie algebra $G$.

In that case, one can rewrite (4.3) as follows:

$$\hat{F}(B, B^\dagger) \mathcal{Z} = 0 \tag{4.6}$$

where $B^\dagger = \frac{\delta}{\delta B}$, $F = dB^\dagger + [B, B^\dagger] - K \cdot B$.

Now let us return to the case of our interest - theory with a holomorphic symmetry. This symmetry is a kind of intermediate between the local (gauge) symmetry, where the parameter of transformation is allowed to depend on the space-time coordinates and the global symmetry where the parameters of transformations are constants. Therefore, in defining the currents and deriving their properties one has some peculiarities, which we are going to explore.

First of all, let us imitate the Noether method of constructing the current: we apply a general transformation with a parameter $\epsilon(x)$ which depends on the space-time coordinate and extract the variation of the action in the linear approximation:

$$\delta_\epsilon S = \int_{\Sigma} \text{Tr}(\delta \epsilon J) \tag{4.7}$$

where we have put $\delta \epsilon$ unlike $d\epsilon$ in equation (4.3) since for holomorphic $\epsilon$ the action shouldn’t change. Certainly, there is a subtlety in this argument, since on a compact manifold there are only constant holomorphic functions, so there is no distinction between holomorphic and global symmetry. On the other hand, if we allow $\Sigma$ to have a boundary, then the action can transform with a boundary terms. So we prefer to work backwards and define holomorphic symmetry as the one where the equation (4.7) is satisfied. The next catch is that $J$ is not uniquely determined from (4.7), since any $J' = J + \delta \lambda$ will work as well (it doesn’t occur in two dimensions, for $J$ is a $(1, 0)$ form). In some cases this freedom can

44
be fixed but one should be aware of it. In fact, even in the classical case one can add to the current $J$ any exact form $d\lambda$. Therefore, the functional $Z$ is not very well defined. Different theories with the same symmetry group differ essentially by the choice of $J$ in the formula (4.2). Holomorphic theories in the dimension $2d$ are distinguished by the fact, that the current $J$ is $(d, d - 1)$ form (generally it would have both components).

Definition of $Z(\bar{A})$ and its equivariance properties. We consider the generating function for the correlators of the currents $J$ in the WZW theory:

$$Z(\bar{A}) = \int Dg e^{-S_\omega(g)} e^{-\frac{i}{\hbar} \int_\Sigma \text{Tr} J \bar{A}}$$

(4.8)

The generating parameter $\bar{A}$ is a $\mathfrak{g} \otimes \mathbb{C}$-valued $(0, 1)$ form on $\Sigma$. \(^{11}\)

Leaving aside the question of the regularization of the quantity on the right hand side of (4.8) we observe the following equivariance property of $Z$, following from the PW formula:

$$Z(\bar{A} h^{-1}) = e^{S_\omega(h) + \frac{i}{\hbar} \int_\Sigma \omega \wedge h^{-1} \partial h \bar{A}} Z(\bar{A})$$

(4.9)

Taking $h$ to be infinitesimally close to 1 we get the following functional equation on $Z$:

$$\hat{F}^{(1, 1) +} Z = \omega \wedge F(\Lambda \frac{\delta}{\delta A}, \bar{A}) Z = 0$$

(4.10)

where

$$F(A, \bar{A}) = \partial \bar{A} - \bar{\partial} A + [A, \bar{A}]$$

and $\Lambda$ is an operation inverse to multiplication by $\omega$. Namely, the action of $\Lambda$ on a three form $\Omega_{\mu \nu \lambda}$ gives a one form

$$\Lambda \Omega = \frac{1}{2} \omega^{\nu \lambda} \Omega_{\mu \nu \lambda},$$

where the bivector $\omega^{\mu \lambda}$ is inverse to $\omega_{\mu \lambda}$. Moreover, since $J$ satisfies the flatness condition:

$$\partial (\Lambda J) + (\Lambda J)^2 = 0,$$

(4.11)

$Z$ obeys

$$\hat{F}^{2, 0} Z = F^{2, 0}(\Lambda \frac{\delta}{\delta A}) Z = 0$$

(4.12)

\(^{11}\) $\bar{A}$ is actually a connection, but we will work on a topologically trivial bundle until section 4.3 below.
4.3. Non-trivial bundles and Čech approach

4.3.1. Moduli of Vector Bundles and ASD Connections

In two dimensions one of the possible ways of defining operators in the theory is through coupling the basic field $g$ to non-trivial gauge fields. For example, in the abelian theory, by coupling to a line bundle $\mathcal{L}$ with non-trivial $c_1$ one can get insertions of operators at the points corresponding to the divisor of $\mathcal{L}$. This procedure can be generalized to the nonabelian case. Therefore, it is natural to try to generalize the $WZW_4$ action for the theory in the background of a non-trivial gauge field.

Let $E \to \Sigma_4$ be a rank $r$ complex hermitian vector bundle with metric $(\cdot,\cdot)_E$ and Chern classes $c_1, c_2$ and let $g(x)$ be a section of $\text{Aut}(E)$. To write a well-defined action generalizing (2.4) we must introduce a connection $\nabla$ on $E$. It gives rise to a connection on $\text{Aut}(E)$, which we also denote as $\nabla$. Since $\Sigma_4$ is complex we can split the connection into its $(1,0)$ and $(0,1)$ pieces: $\nabla = \nabla^{(1,0)} + \nabla^{(0,1)}$. The $WZW_4$ action in the background $(E, \nabla)$ is (we use here the result of [42]):

$$
S_{\omega;E,\nabla}[g] = -\frac{i}{4\pi} \int_{\Sigma_4} \omega \wedge \text{Tr}(g^{-1} \nabla^{(1,0)} g \wedge g^{-1} \nabla^{(0,1)} g) + \\
+ \frac{i}{12\pi} \int_{X_5} \omega \wedge \left[ \text{Tr}(\hat{g}^{-1} \hat{\nabla} \hat{g})^3 + 3 \text{Tr} \, F_{\nabla}[\hat{g}^{-1} \hat{\nabla} \hat{g} + (\hat{\nabla} \hat{g})\hat{g}^{-1}] \right].
$$

(4.13)

In the formula we trivially extended $E$ to $X_5$ and we denote the extension of $g$, $\nabla$ to $X_5$ as $\hat{g}$, $\hat{\nabla}$.

In order for the Polyakov-Wiegmann formula to be generalizable for non-trivial $E$ we must check whether $S_{\omega;E,\nabla}[gh] - S_{\omega;E,\nabla}[g] - S_{\omega;E,\nabla}[h]$ is local in four dimensions. After a simple computation one obtains:

$$
S_{\omega;E,\nabla}[gh] = S_{\omega;E,\nabla}[g] + S_{\omega;E,\nabla}[h] - \frac{i}{2\pi} \int_{\Sigma_4} \omega \wedge \text{Tr}[g^{-1}(\nabla^{(1,0)} g)(\nabla^{(0,1)} h) h^{-1}]
$$

(4.14)

As before we define

$$
\mathcal{A}^{1,1}(E \to \Sigma_4) \equiv \{ \nabla : F^{0,2} = F^{2,0} = 0 \}
$$

(4.15)

to be the space of unitary connections whose curvature is of type $(1,1)$. If $\nabla \in \mathcal{A}^{1,1}$ then $\nabla^{(0,1)}$ is the Dolbeault operator defining an integrable holomorphic structure $\mathcal{E}$ on
the vector bundle $E$. The moduli space of holomorphic vector bundles \(^{12}\) is given by: 
\[ \mathcal{HB} \equiv \{ \nabla^{(0,1)} = \nabla^{(0,1)}_0 + A : (\nabla^{(0,1)})^2 = 0 \} / \text{Aut}(E) \], where $\nabla^{(0,1)}_0$ is some reference connection and $A$ is a $(0,1)$-form with values in $\text{ad}(E)$.

The Ward identities of the bc system from the section 3.5.2, coupled to the nontrivial bundle $E$ can be represented by the identities in the formal functional integral over the group of unitary automorphisms $\text{Aut}(E)^u$ of the bundle $E$. Once again we consider the generator of current correlation functions. For $\bar{A} \in \Omega^{0,1}(\text{ad}(E))$:

\[ Z_E[\bar{A}] = \left\langle \exp \frac{\text{i}}{2\pi} \int_{\Sigma} \omega \wedge \text{Tr} \bar{A} g^{-1} \nabla^{(1,0)}_0 \right\rangle \] \hspace{1cm} (4.16)

By the generalized PW formula (4.14) we have the equivariance condition:

\[ Z_E(\bar{A}^{h^{-1}}) = e^{S_{\omega \cdot E, \nabla^{(0,1)}_0} + \frac{1}{8\pi} \int_{\Sigma} \omega \wedge \text{Tr} \bar{A} h^{-1} \nabla^{(1,0)}_0 h} Z_E(\bar{A}) \] \hspace{1cm} (4.17)

where $\bar{A}^{h^{-1}} = h \bar{A} h^{-1} - (\nabla^{(0,1)}_0 h) h^{-1}$. Restricting $\nabla^{(0,1)}_0 + \bar{A}$ to be in $\mathcal{A}^{1,1}(E \rightarrow \Sigma)$ we learn that $Z_E[\bar{A}] \in H^0(\mathcal{HB}; L_{\omega})$ where the line bundle $L_{\omega}$ has non-trivial first Chern class $c_1(L_{\omega}) = \omega_{\mathcal{M}}$, associated with the Kähler form on $\mathcal{A}^{1,1}(E \rightarrow \Sigma)$; $\omega_{\mathcal{M}} = \int_{\Sigma} \omega_S \wedge \text{Tr} \delta A \wedge \delta A$. In other words, $Z_E[\bar{A}]$ is a wavefunction for the quantization of $\mathcal{HB}$.

Again, the quantization can also be understood as the quantization of the moduli space $\mathcal{M}^+$ of ASD connections. Mathematically, this follows from the Donaldson-Uhlenbeck-Yau theorem \(^{13}\). This theorem states that (under appropriate $\omega$-stability assumptions) there is a complex gauge transformation making the associated unitary connection self-dual $[32]$, thus providing an identification of $\mathcal{HB}$ with $\mathcal{M}^+ = \{ \nabla \in \mathcal{A}[E] : F^+ = 0 \} / \text{Aut}(E)^u$, where $\mathcal{A}[E]$ is the space of unitary connections on $E$. Physically, the relation is provided by the quantum equations of motion for $g$ in (4.16):

\[ \omega \wedge \left[ \nabla^{(0,1)}_0 (g^{-1} \nabla^{(1,0)}_0 g) - \nabla^{(1,0)}_0 \bar{A} + [g^{-1} \nabla^{(1,0)}_0 g, \bar{A}] \right] = 0 \] \hspace{1cm} (4.18)

We view (4.18) as the $(1,1)$ part of the ASD Yang-Mills equation $\hat{F}^+ = 0$.

Remark. We speculate that the holomorphic blocks of nontrivial vertex operator correlators associated to divisors can be obtained by quantization of moduli spaces introduced by

\(^{12}\) Technically one should specify carefully a compactification of this moduli space. One reasonable possibility is to choose the moduli space of rank $r$ torsion free sheaves on $\Sigma$ semistable with respect to a polarization $H$.

\(^{13}\) which plays the role in four dimensions of the Narasimhan-Seshadri theorem.
Kronheimer and Mrowka [47]. These are moduli spaces of ASD connections with singular-
ities along a divisor $D$ such that the limit holonomy around $D$ exists and takes values in
a fixed conjugacy class.

We can also present a formula for the generalized WZW action in any number of
dimensions for the non-trivial background bundle for the hermitian field $h = e^{2i\phi}$ (it
transforms across the patches as $h_\alpha = g_{\alpha\beta} h_\beta g_\alpha^\dagger$). In the unitary frame the transition functions are unitary and therefore the form

$$\omega_\alpha = \int_0^1 dt Tr \phi_\alpha (\bar{\partial}_\lambda e^{-2it\phi_\alpha} \partial_\lambda e^{2it\phi_\alpha} ) \omega$$

is gauge invariant and $\omega_\alpha = \omega_\beta$ on the intersection $U_{\alpha\beta}$. One can present yet another
variant of this formula, using the gauge WZ terms. Some of them were found in [42], [19].

4.3.2. Čech approach

In this section we will try to apply the Čech description of the holomorphic bundles to the study of the holomorphic blocks. First, we start with the case of trivial vector bundle $E$ (it needs not to be trivial as a holomorphic bundle). Given a triangulation of $\Sigma$ as $\bigcup U_\alpha$ we choose the new local trivializations $g_\alpha$ such that the generating parameter $\bar{A}$ of the
generating functional $Z$ equals $(\bar{\partial}_\alpha g_\alpha^{-1}$ on $U_\alpha$. If we multiply $g_\alpha$ on the right by the holomorphic in $U_\alpha$ function $h_\alpha$ then $\bar{A}$ does not change. Now we rewrite the equivariance property of $Z$ as follows:

$$Z(\{g_\alpha\}) = \exp(\sum_\alpha \int_{U_\alpha} \mathcal{L}(g) + \text{Tr}(g^{-1} \partial g \bar{d} g_\alpha g_\alpha^{-1}) \wedge \omega) Z(\{g_\alpha\})$$

Introduce a new functional

$$\Psi(\{g_\alpha\}) = Z(\{g_\alpha\}) \exp(\sum_\alpha \int_{U_\alpha} \mathcal{L}(g_\alpha))$$

Then

$$\Psi(\{g_\alpha\}) = \exp(\sum_\alpha \int_{\partial U_\alpha \times I} \omega \wedge \text{Tr}(\bar{d} g \bar{d} g_\alpha g_\alpha^{-1}) \Psi(\{g_\alpha\})$$

It is clear that $\Psi(\{g_\alpha\})$ depends only on the values of $g_\alpha$ at the boundaries $\partial U_\alpha$ (and their continuation to $\partial U_\alpha \times I$) but not on the values of $g_\alpha$ in the bulk.

On the other hand, the previous invariance under the multiplications of $g_\alpha$ from the
right by the holomorphic in $U_\alpha$ functions $h_\alpha$ is lost. One has a transformation law instead:

$$\Psi(\{g_\alpha h_\alpha\}) = \exp(\sum_\alpha \int_{U_\alpha} \mathcal{L}(h_\alpha)) + \int_{\partial U_\alpha \times I} \text{Tr}(\bar{d} g_\alpha h_\alpha \wedge d h_\alpha h^{-1} \wedge \omega) \Psi(\{g_\alpha\})$$

This is Čech description of the holomorphic blocks. One can refine it using the triangulation. In this way one gets the generalization of the fusion rules.

48
4.4. Alternative descriptions of holomorphic blocks

4.4.1. ADHM approach

For special manifolds \( \Sigma \) there is another way of parameterizing the moduli space of bundles. We describe for simplicity the \( \Sigma = \mathbb{R}^4 \) case, and the ADHM construction for ALE manifolds will be presented in the section 7.4. The generalization to this case is immediate. The ADHM construction for \( \mathbb{R}^4 \) goes as follows. Introduce vector spaces \( V, W \) and a space of operators:

\[
B_{0,1} : V \to V \\
I : W \to V, \quad J : V \to W
\]  

(4.19)

The claim is that the moduli space \( \mathcal{M} \) of \( U(W) \) instantons on \( \mathbb{R}^4 \) with appropriate asymptotics at infinity and instanton charge \( \text{dim}V \) is a quotient of the subspace of operators, satisfying the conditions

\[
[B_0, B_1] + IJ = 0 \\
[B_0, B_1^\dagger] + [B_1^\dagger, B_1] + II^\dagger - J^\dagger J = 0
\]  

(4.20)

by the action of the unitary group \( U(V) \):

\[
B_i \to g^{-1}B_ig, \\
I \to g^{-1}I, \quad J \to Jg
\]  

(4.21)

Another claim is that the symplectic form on \( \mathcal{M} \) we are interested in comes from the form on the ADHM data:

\[
-i\Omega = k \text{Tr} \left[ \delta B_0 \wedge \delta B_0^\dagger + \delta B_1^\dagger \wedge \delta B_1 + \delta I \wedge \delta I^\dagger + \delta J^\dagger \wedge \delta J \right]
\]

The quantization of \( \mathcal{M} \) can be carried out quite explicitly now, we are interested in the holomorphic functions of the ADHM data, which are defined on the submanifold

\[
[B_0, B_1] + IJ = 0
\]

which the following transformation property:

\[
\Psi(g^{-1}B_ig, g^{-1}I, Jg) = (\text{det}g)^k \Psi(B_i, I, J)
\]

It is easy to describe such functionals - they are polynomials of the degree \( k \). Unfortunately, due to non-compactness of \( \mathbb{R}^4 \) the number of such functionals is infinite.
4.4.2. $K3$ case

For the special $K3$ surfaces $\Sigma$ one can parameterize the holomorphic bundles by the collections of points on $\Sigma$ [48]. Namely, for special $K3$ for $G = SU(2)$ for the odd instanton charge $k$ the moduli space is isomorphic to the space of configurations of $2k - 3$ points on $K3$. This space is singular, as points can collide, so the actual moduli space is a hyperkähler resolution of the diagonals in $S^{2k-3}K3$. Then line bundle $L$ over $M$ comes from the line bundle over $K3$, whose first Chern class is a Kähler form $\omega$. The sections of the bundle over $M$ are the symmetric sections over $K3 \times \ldots \times K3$ with some specific behaviour near diagonals.

The similar statement holds for genus one conformal blocks in two dimensional $WZW$ theory for $G = SU(N)$. There, the moduli space of bundles over the elliptic curve is a configuration space of $N$ points on the torus with fixed center of mass. It would be nice if this analogy actually helps studying the blocks in four-dimensional problem.

4.4.3. Abelian variety

For the trivial instanton sector for $\Sigma = T^4$ the holomorphic bundles are characterized by the points in the dual torus. The holomorphic blocks for the avatar of $WZW_2$ are given by the sums of products of theta-functions, corresponding to the principal polarization $\omega$. Indeed, just like in two dimensional case, the generic bundle splits into a sum of line bundles. The holomorphic block is invariant under the exchange of the summands in the decomposition:

$$E = \bigoplus_i L_i$$

It imposes some restrictions on the behaviour of the products near the diagonals.

For the generalized $WZW$ theory in the abelian case it is easy to prove that there are no non-trivial holomorphic blocks. It has to do with the degeneracy of the form $F \delta A \delta A$ on the space of flat connections.

4.5. Projected theories

In this section we let $\Sigma_4$ be an algebraic surface, and denote it as $S$. Suppose we have a holomorphic map

$$f : S \to C$$

where $C$ is an algebraic curve. Correspondingly, there are maps of the moduli spaces, line bundles and their cohomologies.
\[ f^*: \mathcal{H}B(C) \to \mathcal{H}B(S) \]
\[ (f^*)^*: \mathcal{L} \to L^{\otimes k} \]  \hspace{1cm} (4.23)
\[ (f^*)^*: H^i(\mathcal{H}B(S), \mathcal{L}) \to H^i(\mathcal{H}B(C), L^{\otimes k}) \]

Here \( L \) is the standard determinant line bundle over \( \mathcal{H}B(C) \), and \( k \) equals the integral of \( \omega \) over a generic fiber of \( f \):
\[ k = \int_{f^{-1}(x)} \omega \]  \hspace{1cm} (4.24)

This simple observation can be interpreted in more physical terms as follows: Let us define the “projected current” to be:
\[ J(x) dx \equiv \int_{f^{-1}(x)} \omega \wedge g^{-1} \partial g = f^* (\omega \wedge g^{-1} \partial g) \]  \hspace{1cm} (4.25)

This is a holomorphic 1-form on \( C \) and defines an ordinary two-dimensional current algebra with central charge \( k = f^* \omega \). Therefore we can proceed to use all the standard constructions from 2D RCFT. For example, we can construct chiral vertex operators which have all the familiar properties of anomalous dimensions, monodromy representations of the braid group, etc.

Remarks.

1. In 2D RCFT chiral vertex operators are sometimes heuristically regarded as path-ordered exponentials of a chiral current. In the present context such an identification would read as follows. Let \( \Gamma \) be a curve in \( C \) connecting two points \( x_0 \) and \( x_1 \). Then, given a representation \( \rho_\lambda \) of the group consider the formal relation:
\[ \rho_\lambda \left( P \exp \int_{\Gamma} dx \left\{ \int_{f^{-1}(x)} \omega \wedge g^{-1} \partial g \right\} \right) = V_\lambda[D]V_{\lambda'}[D'] \]  \hspace{1cm} (4.26)
\[ D = f^{-1}(x_0), D' = f^{-1}(x_1) \]

which might be expected to define 4D chiral vertex operators. In fact, it is a nontrivial problem to define a proper regularization of this expression. \(^{14}\) From this point of view, anomalous dimensions arise because the projected Green function has logarithmic singularities.

\(^{14}\) Similarly, the proper quantum definition of a Wilson loop in a nonabelian gauge theory is a highly nontrivial problem.

51
2. It is natural to ask what kind of moduli space vertex operators such as those discussed above might correspond to. In a two dimensional theory the insertion of the vertex operator at a point \( x \) corresponds to the moduli space of holomorphic bundles \( \mathcal{E} \) with parabolic structure at the point \( x \), i.e. there is a flag of subspaces in the fiber \( \mathcal{E}|_x \) over the point. The pullback of this bundle to \( S \) will produce a holomorphic bundle with parabolic structure along the divisor \( f^{-1}(x) \). In this way we again relate correlation functions to the quantization of Kronheimer-Mrowka moduli spaces, since these may be identified with moduli spaces of holomorphic vector bundles with parabolic structure along a divisor \([49][50]\).

4.6. Hecke operations

The discussion of the previous section makes contact with some aspects of the Beilinson-Drinfeld “geometric Langlands program” and also with some constructions of Nakajima. Indeed, if \( h \partial_c h^{-1} = \tilde{\partial}_c \) then the holomorphic vector bundles are isomorphic away from the singularities of \( h \), thus, under good conditions, they will fit into a sequence:

\[
0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to S \to 0 \quad (4.27)
\]

for some sheaf \( S \). Conversely, given \( S \) we can form the “geometric Hecke correspondence:”

\[
\mathcal{H}B(c_1, c_2) \xrightarrow{\pi_1} \mathcal{P}_S \xleftarrow{\pi_2} \mathcal{H}B(c_1', c_2') \quad (4.28)
\]

defining, in a standard way, linear operators \((\pi_1)_*(\pi_2)^*\) and \((\pi_2)_*(\pi_1)^*\) on the cohomology spaces. This construction appears in the work of Beilinson and Drinfeld. See also \([51][52][53][54]\).

One example of this construction is familiar in CFT from the study 2D Weyl fermions \([7]\). The correspondence analogous to (4.28) defined by the sequences

\[
0 \to \mathcal{L} \otimes \mathcal{O}(-P) \to \mathcal{L} \to \mathcal{L} |_{P \to 0}
\]

\[
0 \to \mathcal{L} \to \mathcal{L} \otimes \mathcal{O}(Q) \to [\mathcal{L} \otimes \mathcal{O}(Q)] |_{Q \to 0}
\]

(4.29)

correspond to the effect of inserting \( \psi(P), \bar{\psi}(P) \) respectively in the correlation functions which define sections of \( H^0(\mathcal{H}B(c_1); \text{DET}\partial_{\mathcal{L}}) \).

It seems natural to conjecture that if \( h \) is singular along divisors or at points then the operators \((\pi_1)_*(\pi_2)^*\) are equivalent to insertion of the vertex operators discussed in sections 4.3, 4.5.
Nakajima’s algebras. In [51] Nakajima has shown that, using the ADHM construction of $U(k)$ gauge theory instantons on ALE spaces $X_n = \mathcal{C}^2/\mathbb{Z}_n$ one can construct highest weight representations of affine Lie algebras. More precisely, connections on $U(k)$ vector bundles $E \to X_n$ are specified by $c_1(E) \in \Lambda_{root}(SU(n))$, $[55]_2(E) \in \mathbb{Z}_+$, and a flat connection at infinity, i.e., a flat connection on the Lens space $S^3/\mathbb{Z}_n$. The McKay correspondence gives a 1-1 equivalence between flat $U(k)$ connections on $S^3/\mathbb{Z}_n$ and integrable highest weight representations of $\widehat{SU(n)}_k$ at level $k$. Let us denote this correspondence as $\rho(\lambda) = \sum \ell_i \rho_i \in Hom(\mathbb{Z}_n \to U(k))/U(k) \leftrightarrow \lambda(\rho) \in HWT(\widehat{SU(n)}_k)$ where $\ell_i$ are nonnegative integers assigned to the nodes of the extended Dynkin diagram and $\sum \ell_i = k$.

Nakajima’s theorem states that the cohomology of the moduli spaces of “ASD instantons” on $X_n$ is a representation of $\widehat{SU(n)}_k$ at level $k$. The reason for the quotation marks is explained in the following section. Moreover, $H^p(M(c_1,[55]_2,\rho(\lambda)))$ where $p = \frac{1}{2} \dim M(c_1,[55]_2,\rho(\lambda))$, is naturally identified with the weight space $\hat{p} = c_1, L_0 = [55]_2(E)$ of the representation $\lambda(\rho)$ of $\widehat{SU(n)}_k$. Nakajima proves this statement using the ADHM construction of instantons. The generators of the affine Lie algebra are defined using a geometric Hecke correspondence, as described above.

4.7. Algebraic sector

In this section we assume that the quantum theory can be defined preserving invariance of the measure $Dg$.

From the PW identity we get

$$\left\langle \exp\left[\frac{i}{2\pi} \int_{\Sigma_4} \omega \wedge \text{Tr}[g^{-1}\partial g \partial h h^{-1}] \right]\right\rangle = \exp\left[S_\omega[h]\right] \quad (4.30)$$

This identity is closely related to the algebraic geometry of holomorphic vector bundles on $\Sigma_4$ and characterizes the content of the algebraic sector of the theory. In the following subsections we will extract some interesting information from this general statement.

Recall that $\mathcal{Z}(\overline{A})$ satisfies the two functional equations (4.10) and (4.12). Unfortunately, in the non-abelian theory the equation $\hat{F}^{2,0} \mathcal{Z} = 0$ is non-linear and is apparently insoluble. Therefore, we shall restrict $\mathcal{Z}$ to the space of connections $\overline{A}$ such that $F^{0,2}(\overline{A}) = 0$. (Such connections will be called Kähler gauge fields and the space of such connections is denoted as $\mathcal{A}^{1,1}$.)

Our definition of the algebraic sector of WZW$_4$ theory is as follows: it is the set of correlators, which can be extracted from the properties of $\mathcal{Z}(\overline{A})$, evaluated on $\mathcal{A}^{1,1}$.
Naively, one can argue that the solution of the equation $\hat{F}^{2,0}Z = 0$ is uniquely determined by the restriction of $Z$ onto the subspace $A^{1,1}$, and thus all current correlators are determined by the algebraic sector.

For the generalized WZW theory one may define

$$Z(\bar{A}, \bar{a}) = \left\langle \exp i \int_{\Sigma} \text{Tr}(\bar{A}J + \bar{a}j) \right\rangle$$

for $j = g^{-1} \partial g, J = j \bar{\partial} j$.

The Definition of the algebraic sector in the generalized WZW case is the following: it is the set of correlators, which one can get from the properties of $Z(\bar{A}, \bar{a})$, where $\bar{A}$ is $(0,1)$ component of the connection form and $\bar{a}$ is a $(1,2)$ form with values in the adjoint, restricted onto subspace, where

$$\bar{a} = \left( \bar{A} \partial \bar{A} \right)_{\mathfrak{g}}$$

Here $()_{\mathfrak{g}}$ means the projection onto the Lie algebra $\mathfrak{g}$.

Simple examples of algebraic correlators: "Divisor current" correlators In WZW$_2$ we study correlators of currents at points $z_1, \ldots, z_n$, i.e.

$$\langle j(z_1) \cdots j(z_n) \rangle.$$ 

Note that points are divisors in one dimensional complex geometry. For example, on $\mathbb{C}\mathbb{P}^1$ a point $z_1$ is the polar divisor of the function $f_1(z) = 1/(z - z_1)$, and thus,

$$j(z_1) = \int_{Y_1} j(z) f_1(z),$$

where $Y_1$ is a small circle around the point $z_1$. As in (3.28)(3.29) this observable generates gauge transformations and hence its correlators are easily computed on $\mathbb{C}\mathbb{P}^1$. Equivalently, the generating function of current correlators is a partition function in the presence of a field $\bar{A}$. This field defines the structure of a holomorphic vector bundle on an algebraic curve. If the bundle is holomorphically trivial we may find the correlators from the PW formula. Even if the bundle is nontrivial, but the curve is $\mathbb{C}\mathbb{P}^1$ we may use singular gauge transformations (discussed below) to find the correlators. If the genus of the curve is greater than zero, the PW formula reduces the problem of the computation of current correlators to the problem of the quantization of the moduli space of holomorphic vector bundles.

54
In complex dimension two, arbitrary correlators of currents are generated by the function \( Z(\bar{A}) \) for unrestricted \( \bar{A} \), and hence cannot be computed within the algebraic sector. Nevertheless, it is possible to use the current to define an observable at a divisor \( D \) by analogy with the previous case, i.e. consider a meromorphic function \( f \in \mathfrak{g} \otimes \mathbb{C} \) on \( \Sigma_D \) such that its polar divisor is \( D \). Consider the three-dimensional boundary \( Y \) of a small tubular neighborhood around the divisor. Let us define an observable \( J(f, D) \), associated to \( (f, D) \) as

\[
J(f, D) = \int_Y \text{Tr}(f J)
\]

This is just (3.28) above, and only depends on \( f, D \).

Correlators of the observables (4.32) are generated by connections \( \bar{A} \in \mathcal{A}_{11}^{11} \), and are therefore in the algebraic sector. Moreover, correlators of the observables (4.32) can be computed as follows. Suppose that the divisors \( D_i \) do not intersect one another so that their tubular neighborhoods do not intersect. Each observable generates holomorphic gauge transformations outside its tubular neighborhood and since the current transforms like a connection the "2-divisor" correlator is:

\[
\left< J(f_1, D_1)J(f_2, D_2) \right> = \int_{Y_2} \text{Tr}\left(f_2 \omega \wedge \partial f_1 \right)
\]

Similarly, the "3-divisor" correlator is:

\[
\left< J(f_1, D_1)J(f_2, D_2)J(f_3, D_3) \right> = \int_{Y_2} \text{Tr}\left\{ f_3 \omega \wedge \partial \left( [f_1, f_2] \right) \right\} + \int_{Y_3} \text{Tr}\left\{ f_3 \omega \wedge \partial \left( [f_3, f_1] \right) \right\}
\]

The formulae (4.33)(4.34) generalize current algebra on algebraic curves to current algebra on algebraic surfaces. Similarly the \( n \)-divisor correlators can be written as above.

Remark. One could also define observables analogous to (4.32) associated to arbitrary divisors, rather than polar divisors of meromorphic functions. These are still in the algebraic sector but their correlators cannot be computed as easily since they require information about nontrivial conformal blocks. \(^{15}\)

Algebraic Correlators in the Abelian Theory. In some two-dimensional conformal field theories one can compute not only current correlators but also vertex operator correlators

\(^{15}\) Compare with the current correlator on a Riemann surface of genus \( g > 0 \) in \( WZW_2 \).
using singular gauge transformations. In the 4D $U(1)$ $WZW_4$ theory (which is just a Gaussian model) this method may be generalized as follows.

We would like to make a singular complexified gauge transformation by a meromorphic function $f$:

$$g = e^{i\phi} \rightarrow fgf^\dagger$$

(4.35)

Intuitively, along the zero and polar divisors of $f$ the effect of the singular gauge transformation is to insert a vertex operator. We can make this idea precise and compute the correlators using the (left- and right-) PW formula by using a regularized version of $f$. Suppose $\text{div}(f) = \sum n_i D_i$, and, near $D_i$, we may describe the divisor in local coordinates as $z_i = 0$. Thus we have

$$f = z_i^{n_i} (f_i + O(z_i))$$

(4.36)

where $f_i$ can be meromorphic on $D_i$. We now use the K"ahler metric to choose tubular neighborhoods $T_i$ around $D_i$ of radius $\epsilon_i$, and use a partition of unity to define a regularized gauge transformation with parameter $f_\epsilon$ such that $|f_\epsilon|^2 = |f|^2$ outside $\bigcup T_i$, while $|f_\epsilon|^2 = (|z_i|^2 + \epsilon_i^2)^{n_i} |f_i + \cdots|^2$ in $T_i \setminus \bigcup_{j \neq i} T_j$ etc.

Now consider the effect of such a gauge transformation, in the left-right version of the PW formula (4.30). The LHS of this formula involves a change of action by:

$$S_\omega \rightarrow S_\omega + \sum_i \left( i n_i \int_{D_i} \omega \phi + O(\epsilon) \right)$$

(4.37)

as $\epsilon \rightarrow 0$. So the LHS formally becomes a correlator of operators

$$V_k[D] = e^{i k \int_D \omega \phi}$$

(4.38)

Computing the RHS of this formula we find, as $\epsilon \rightarrow 0$

$$S_\omega [\log |f_\epsilon|^2] \rightarrow \frac{1}{2} \sum_i [n_i^2 \log \epsilon_i^2 \int_{D_i} \omega + n_i \int_{D_i} \omega \log |f_i|^2]$$

(4.39)

\footnote{See, for example [7][6][13]. In [13] these relations were called “multiplicative Ward identities.”}

\footnote{These remarks are closely related to ref. [56]. Another related example is given by Maxwell theory on $\Sigma_4$. Singular gauge transformations shift the line bundle and allow one to produce a classical partition function analogous to the partition functions of 2d gaussian models. They also allow one to study the vertex operators $V(D) = \exp \left[ \frac{i}{2\pi} \int_D F \right]$ and $V^*(D) = \exp \left[ \frac{i}{2\pi} \int_D *F \right]$. This and related issues have been independently studied in [57][58].}
Exactly as in two dimensions the logarithmically divergent terms are renormalization factors needed to normal-order the observables (4.38). These observables accordingly have anomalous dimension

\[ \Delta_k = \frac{k^2}{2} \int_D \omega \]  

and the renormalized operators have the correlation function:

\[ \left\langle \prod_i V_{n_i}[D_i] \right\rangle = \exp \left[ \sum_i \frac{n_i}{2} \int_{D_i} \omega \log |f_i|^2 \right] \]  

**Remark.** We expect that the correlator will factorize holomorphically, as in two-dimensions, and that it should be possible to define “chiral vertex operators.” One may expect a relation like:

\[ \exp \left[ \int_{D_0} \text{ord}_{D_0}(f) \omega \phi_L \right] \exp \left[ \int_{D_\infty} \text{ord}_{D_\infty}(f) \omega \phi_L \right] \sim \exp \int_T \omega \wedge \partial \phi \]  

where the zero and polar divisors form the boundary of a three-manifold \( T \): \( D_0 - D_\infty = \partial T \). The details of this proposal, especially in the nonabelian case, appear to be nontrivial and have not been worked out yet.

**Algebraic correlators in generalized abelian WZW theory.** We shall briefly discuss the generalization of the previous results to the theory \( \partial \phi \bar{\partial} \phi \bar{\partial} \phi \). As we discussed at the end of the section 3.4 the divisor operator must be accompanied by the point-like operator inserted at the self-intersection:

\[ \mathcal{V}(D) \sim \exp \left[ \int_{\Gamma^3} \partial \phi \bar{\partial} \phi + \int_{\Gamma^1} \partial \phi \right] \]

where \( \partial \Gamma^3 = D \), \( \partial \Gamma^1 = D \cdot D \). We assume, that \( D \) is a divisor of some meromorphic function \( f \) and \( \Gamma_i \) are counted with multiplicities, according to the order of the pole or zero of \( f \). One can always find such \( \Gamma^3 \) since \( D \) is trivial in homology. Let \( D_i \) be the irreducible components of \( D \) and let \( f \) behave near \( D_i \) as follows:

\[ f = z_i^{n_i} (f_i + o(1)) \]

where \( z_i \) is a local coordinate around \( D_i \), such that \( z_i = 0 \) is the equation, which defines \( D_i \). Near the intersection (we assume it is normal crossing) \( D_i \cap D_j \) we have \( f = z_i^{n_i} z_j^{n_j} (f_{ij} + o(1)) \). Then the chiral part of the expectation value of \( \mathcal{V}(D) \) is given by\(^\text{18}\):

\[ \exp i \sum_{i \neq j} n_i n_j \log f_{ij}(D_i \cap D_j) \]

\(^{18}\) The formulas of this type were proposed by A. Gerasimov

57
4.8. Appendix. Holomorphic bundles and their moduli

Recall that a (locally trivial) vector bundle $E$ over a space $B$ (called a base) is a space $E$ with a map (projection) $p : E \to B$ such for any disc $U_{\alpha}$ in $B$ the restriction $p^{-1}(U_{\alpha})$ of the bundle is trivial, i.e. isomorphic (diffeomorphic or homeomorphic) to the direct product $U_{\alpha} \times F$ where $F$ is a vector space (fiber). The isomorphism $\psi_{\alpha} : p^{-1}(U_{\alpha}) \to U_{\alpha} \times F$ should act fiberwise, i.e. it should intertwine the projection $p$ and the natural projection $p_1 : U_{\alpha} \times F \to U_{\alpha}$. It is called a local trivialization. Over the intersection of two patches $U_{\alpha\beta} = U_{\alpha} \cap U_{\beta}$ one has two trivializations and they define a transition function $g_{\alpha\beta} : U_{\alpha\beta} \to GL(F)$ by the formula: $g_{\alpha\beta} = \psi_{\beta} \circ \psi_{\alpha}^{-1}$.

Now, suppose $B$ is a complex manifold. Then the vector bundle $E$ is holomorphic if the transition functions are holomorphic.

Suppose one has a holomorphic bundle $E$ and chooses another trivialization. Let $\psi_{\alpha}$ be the holomorphic trivialization (also called frame) and $\chi_{\alpha}$ is some trivialization. The following group-valued function on $U_{\alpha}$ is defined:

$$g_{\alpha} = \chi_{\alpha}\psi_{\alpha}^{-1}$$

Over the intersection $U_{\alpha\beta}$ $g_{\alpha}$ and $g_{\beta}$ are related via:

$$g_{\beta} = h_{\alpha\beta}g_{\alpha}g_{\alpha\beta}^{-1}$$

where $h_{\alpha\beta}$ are the holomorphic transition functions and $g_{\alpha\beta}$ are the reference transition functions. Define the $(0,1)$ forms with values in the Lie algebra:

$$\bar{A}_\alpha = g_{\alpha}^{-1}\partial g_{\alpha}$$

Across the patches they transform via the gauge transformation with parameter $g_{\alpha\beta}^{-1}$. It means that $\bar{A}_\alpha$ represent the connection form $\nabla^{(0,1)}$. It satisfies the $(0,2)$-flatness condition $F^{02} = \nabla^{(0,1)^2} = 0$.

Conversely, given a gauge field whose $(0,2)$ part of the curvature vanishes one can construct locally the functions $g_{\alpha}$ and find a holomorphic trivialization $\chi_{\alpha} = g_{\alpha} \circ \psi_{\alpha}$. This holomorphic trivialization would not change if one performs a global gauge transformation on $g_{\alpha}$ and $\psi_{\alpha}$. On the other hand, the gauge field changes by a global gauge transformation. If the holomorphic transition functions are changed by a global holomorphic function as $h_{\alpha\beta} \to h^{-1}h_{\alpha\beta}h$, which is compensated by the change in the trivializations $\chi_{\alpha} \to h\chi_{\alpha}$,
then $\tilde{A}_\alpha$ remains intact. We learn, therefore, that the space $\mathcal{M}$ of equivalence classes of holomorphic structures on a given vector bundle $E$ is in one-to-one correspondence with the space of $(0,1)$ connections $\nabla^{(0,1)}$ which square to zero: $F^{0,2} = 0$, up to gauge transformations. We should stress, though, that the gauge transformations take values in the complexified gauge group, opposed to the usual gauge transformations, which take values in the compact group.

The space $\mathcal{M}$ of parameters of equivalence classes of bundles is called a moduli space of holomorphic bundles. Usually when one takes a quotient with respect to the non-compact groups one has to be careful with the proper choice of the subset of the original space. The classical example is the space $\mathbb{C}^n$ acted on by the group $\mathbb{C}^*$ of the complex numbers, not equal to zero. The group acts by multiplication and the quotient is not separable. One has to remove the point $\{0\}$ and then the quotient is well-defined and can be identified with the complex projective space $\mathbb{C}\mathbb{P}^{n-1}$. To find the exact conditions on the subspace to be taken is the problem of Geometric Invariants Theory. The general problem with taking quotients is the existence of points, where the group does not act freely. If the stabilizer (little group) of the point is bigger than the stabilizer of the generic point then the quotient would have singularities.

For the moduli space of bundles the points (holomorphic structures) with extra symmetries correspond to the so-called unstable bundles. The bundle is referred to as $\omega$-semi-stable if every subbundle $F \subset E$ satisfies the condition

$$\int_\Sigma \omega^{n-1} \wedge (\frac{c_1(F)}{rkF} - \frac{c_1(E)}{rkE}) \leq 0$$

and it is called stable if the strict inequality holds. Here $\omega$ is a Kähler form and $n$ is a complex dimension of $\Sigma$.

In physics the moduli space of holomorphic bundles appears as a moduli space of instantons. In four dimensions the equation $F^{0,2} = 0$ is just one of the three components of the anti-self-duality equations. Another equation is the complex conjugate to this and the last one $F^{1,1} = 0$ is not invariant under the complex gauge transformations. It is invariant under the unitary transformations. Thus, we impose three real equations and take a quotient with respect to a compact group. Before we were imposing one complex equation and taking quotient with respect to a complex group. It seems that these procedures are equivalent.

Indeed, it turns out that precisely semi-stable bundles correspond to instantons. This is the Donaldson-Uhlenbeck-Yau theorem.
Nevertheless, the replacement of the moduli space of bundles by the moduli of instantons doesn’t guarantee the smoothness of the space \( \mathcal{M} \). The problem are the reducible connections.

These are the gauge fields, which have more gauge symmetries then the generic ones. Not every bundle \( E \) can have a reducible connection, in fact, only those bundles, which split as sums can have those. Moreover, if we restrict the gauge field to be anti-self-dual this imposes the condition on a metric on \( \Sigma \).

Consider for simplicity the rank 2 case. If the bundle \( E \) has trivial \( c_1(E) \) (which corresponds to the \( SU(2) \) gauge theory) then it can only decompose as a sum of a line bundle and its inverse:

\[
E = L \oplus L^{-1}
\]

Now the second Chern class of \( E \) must satisfy the condition:

\[
c_2(E) = -c_1(L)^2
\]

Moreover, since the Chern class is represented by a curvature of an instanton field, its DeRham representative belongs to the negative subspace of the intersection form \( <,> \) on \( H_{\mathbb{R}} = H^2(\Sigma, \mathbb{R}) \). On the other hand, it is integral class and therefore the integral lattice \( H_{\mathbb{Z}} = H^2(\Sigma, \mathbb{Z}) \) should lie in \( H_{\mathbb{R}}^{-} \). If \( b_2^+ \geq 1 \) then one can always perturb a metric on \( \Sigma \) in such a way, that this doesn’t occur. Sometimes one has to be sure that the phenomenon of appearing of reducible connections doesn’t take place in one-parameter family of metrics. In that case one needs the condition \( b_2^+ > 1 \). On the Kähler manifold \( b_2^+ = 1 + 2b_2^{2,0} \), since the self-dual forms are precisely the Kähler form and the forms of the type \( (2,0) \) and \( (0,2) \). See [59], [60], [57] for the progress in the case \( b_2^+ \leq 1 \).

Another important problem which occurs when one wants to integrate something over \( \mathcal{M} \) is the compactness. Compactness of the space of instantons can fail due to the zero size instantons. Recall the simplest solution of the ASD equations on \( \mathbb{R}^4 \), as found in [61]:

\[
A^\alpha_\mu = \frac{\eta_{\mu\nu}(x-x_0)^\nu}{(x-x_0)^2 + \rho^2}
\]

\[
F^\alpha_{\mu\nu} = \frac{\eta_{\mu\nu}}{((x-x_0)^2 + \rho^2)^2}
\]

Here \( x_0 \) and \( \rho \) are the center and the radius of the instanton. The moduli space in question is the space of centers and radii. It is non-compact because \( x_0 \) can run off to infinity. This is not a problem since in the case where \( \mathbb{R}^4 \) is replaced by compact \( \Sigma \) the instanton has
nowhere to run. Another problem is that $\rho$ can shrink to zero. It doesn’t cost any action since the action is given by the instanton charge and doesn’t change as one varies $x_0$ or $\rho$. So one has to add $\rho = 0$ to the space $\mathcal{M}$. But the zero size instanton is not quite a gauge field. It is a singular pure gauge.

In the case of higher instanton number $c_2 > 0$ one has more complex phenomena, where some part of the instanton charge is “stored” in the zero size instantons.

4.9. Appendix. Kähler manifolds and twisted supersymmetry

On a Kähler manifold one can split the components of the gauge field $A_\mu dx^\mu = A + \bar{A}$, its fermionic partner $\psi_\mu dx^\mu = \psi + \bar{\psi}$ etc., and the equivariant derivative $\delta$ as a sum of “holomorphic” and ”anti-holomorphic” parts: $\delta = \delta' + \delta''$:

\[
\begin{align*}
\delta' A &= 0 & \delta' \psi &= \partial_A \phi \\
\delta'' A &= \psi & \delta'' \psi &= 0 \\
\delta' \bar{A} &= \bar{\psi} & \delta' \bar{\psi} &= 0 \\
\delta'' \bar{A} &= 0 & \delta'' \bar{\psi} &= \bar{\partial}_A \phi \\
\delta' \phi &= 0 & \delta'' \phi &= 0
\end{align*}
\]

(4.43)

It is clear, that $\delta'^2 = \delta''^2 = 0$ and that the anticommutator of $\delta'$ and $\delta''$ is a gauge transformation with the parameter $\phi$. We can introduce a second ”ghost” number $p - q$, so that the fields have the following degrees:

\[
\begin{array}{ccc}
\text{field} & p & q \\
\phi & 1 & 1 \\
A, \bar{A} & 0 & 0 \\
\psi & 0 & 1 \\
\bar{\psi} & 1 & 0 \\
\end{array}
\]

(4.44)

The commutator of $\delta'$ and $\delta''$ is a second-order differential operator, which contains an ‘axial’ gauge transformation (the complex gauge transformation with hermitian generator $i\phi$). We use this fact in the derivation of Quillen anomaly and its higher-dimensional counterparts (see also Appendix to the next chapter, where more conceptual explanation of the derivatives $\delta'$ and $\delta''$ is presented in the framework of equivariant cohomology).

61
5. Topological sector

This chapter is devoted to the studies of the topological theories, related to the spaces of
holomorphic blocks. Recall, that the holomorphic parts of the correlation functions
are not uniquely defined. The holomorphic blocks exhibit a monodromy as one varies the
parameters of the system. The physical correlation functions, though, are the single-valued
functions on the parameter spaces.

The holomorphic blocks are the functionals of the external fields. The space of external
fields (which include the gauge and gravitational backgrounds) carries a natural complex
structure. The holomorphic blocks are the holomorphic functions on this space. The
space of backgrounds is acted on by a symmetry group (gauge group, in particular). The
blocks transform under this action in a particular way. Therefore, the blocks determine
the holomorphic sections of some line bundle over the spaces of gauge orbits.

One of the tasks of the topological theory associated to the holomorphic one is to count
the number of the holomorphic blocks. There are several approaches to this problem, each
of its own significance.

First approach formulates the counting problem as a problem of computing an index
of some \( \bar{\partial} \) operator and by virtue of the Riemann-Roch theorem reduces the question to
some integration over the moduli space of holomorphic bundles. The integrand turns out
to be the combination of the Donaldson observables.

Second approach formulates the problem as a quantization of the moduli space \( \mathcal{M} \) of
instantons. The quantum mechanical system on \( \mathcal{M} \) can be considered as a limit of the
\( N = 2 \) supersymmetric gauge theory on the manifold \( \Sigma \times S^1 \). The theory can be treated in
two ways. One can use the supersymmetry to justify the exactness of the approximation,
in which the circle \( S^1 \) shrinks to zero. In that case the effective \( N = 2 \) supersymmetric
gauge theory on \( \Sigma \) reproduces the integrand of the index theorem. Another approach uses
the relation of the gauge theory and a string theory. Namely, in the compactification of
the heterotic string on \( K3 \times S^1 \) one can get various gauge groups in five dimensions. The
knowledge of the exact geometry of the moduli space of the gauge theory, which recently
emerged from the string duality might be used to get another insight on this problem.

The last approach suggests a generalization of the problem. One can observe a ladder
of complications, escorting the problem of counting the blocks: rational, trigonometric and
elliptic cases.

The simplest thing to do is to evaluate the number in the quasiclassical approximation,
computing only the symplectic volume of \( \mathcal{M} \). In some theories this can be done using the
recent progress in understanding the Donaldson theory due to Seiberg-Witten solution of
low-energy $N = 2$ super-Yang-Mills theory. This problem corresponds to the rational level.
The trigonometric case corresponds to the actual counting and it is called trigonometric
since the answer involves the trigonometric functions (essentially coming from the $A$-genus
form, or, from the gauge theory point of view, from the Vandermonde determinants). The
trigonometric level corresponds to the gauge theory on $\Sigma \times S^1$. The last - elliptic - level has
to do with the elliptic genera of the moduli spaces and with the gauge theory on $\Sigma \times T^2$.

To illustrate this abstract classification on a concrete example, consider three formulas.
For the group $G = SU(2)$ let $j$ be a positive number. Consider a coadjoint orbit $O_j$ of
$\frac{1}{2} \sigma_3$, which is a two-sphere with the symplectic form

$$\Omega \sim j d\varphi \wedge d\cos(\vartheta)$$

Let $P = \frac{ip}{2} \sigma_3$ be the element of the Lie algebra of $SU(2)$. It generates the infinitesimal
rotation of $O_j$ around the north pole $\varphi \to \varphi + pt$, $\vartheta \to \vartheta$. The Hamiltonian of this rotation
is $H_p = j p \cos(\vartheta)$. The first (rational) formula is the equivariant symplectic volume of the
sphere:

$$\frac{1}{2\pi i} \int_{O_j} e^{iH_p} \Omega = \frac{e^{ijp} - e^{-ijp}}{ip}$$

The second (trigonometric) formula is the character of the element $g = e^p$ in the spin $j$
representation of $SU(2)$ (for $j$ positive integer):

$$\chi_j(g) = \frac{e^{i(j+1)p} - e^{-ijp}}{e^{ip} - 1} = \frac{\sin((j + \frac{1}{2})p)}{\sin(\frac{1}{2}p)}$$

The last - elliptic formula is the Kac-Weyl formula for the character of the representation
of the loop group of $SU(2)$, more precisely, of $LSU(2) \times S^1$:

$$\chi_{j,k}(gq^{L_0}) = \frac{q^{-\frac{1}{12} \frac{(j+\frac{1}{2})^2}{k+2}^2}}{D_W(p)D_K(p)} \left[ \Theta(j+\frac{1}{2},k+2)(\frac{p}{2\pi};\tau) - \Theta(j+\frac{1}{2},k+2)(\frac{-p}{2\pi};\tau) \right]$$

where $D_K(p)D_W(p) = q^{-\frac{1}{12} \frac{\bar{\eta}(\bar{g}\tau)}{\eta(\tau)}}$, and

$$\Theta(j+\frac{1}{2},k+2)(\frac{p}{2\pi};\tau) = \sum_{n \in j+\frac{1}{2}+(k+2)\mathbb{Z}} q^{\frac{n^2}{2}} e^{i\pi n}$$

We will make an attempt to find all three structures in the appropriate theories.
5.1. Instanton theta functions

We have argued previously that the important objects which arise in any holomorphic theory on a compact manifold are the holomorphic blocks. Recall that they appeared in the study of the generating function of the current correlators:

\[ Z(\bar{A}) = \langle \exp\left(-\int_{\Sigma} \text{Tr} J \bar{A}\right) \rangle \]

The function \( Z \) satisfies a certain equivariance property:

\[ Z(\bar{A}) = e^{-\sigma(\bar{A})} Z(\bar{A}g^{-1}) \]

which can be reinterpreted as an anomaly equation.

Of course, it is impossible to deduce what \( Z \) is just from this property (unless \( \Sigma \) is two dimensional). So we take a position that algebraic correlators can be computed from the restriction of \( Z \) onto the subspace \( A^{1,1} \), i.e. where \( F_{02} = 0 \). On this space acts the group \( \mathcal{G}_0 \) of complex gauge transformations and the quotient \( \mathcal{M} = A^{1,1}/\mathcal{G}_0 \) is finite-dimensional. The transformation property of \( Z \) implies that it descends to a holomorphic (since it depends on \( \bar{A} \) only) section of some line bundle \( \mathcal{L} \) over the moduli space \( \mathcal{M} \).

In two dimensions the moduli space \( \mathcal{M} \) is essentially the moduli space of flat connections (more precisely, the suitable compactification of the moduli space of holomorphic bundles can be identified with the moduli space of flat connections in the compact group). All the line bundles over this space are the powers of some fundamental one - namely the determinant line bundle (see the chapter 3). The sections of the determinant bundle are called non-abelian theta functions of A.Weil.

In four dimensions the moduli space (again, suitably compactified) can be identified with the moduli space of instantons (actually, anti-self-dual gauge fields). There are several classes of natural line bundles over \( \mathcal{M} \). For example, each algebraic curve \( S \) embedded in \( \Sigma \) determines a line bundle \( \mathcal{L}_S \) over \( \mathcal{M} \) which is a pull-back of the determinant bundle over \( \mathcal{M}_S \) via the restriction map. For the rank higher then 2 there is another line bundle (its existence is correlated to the chiral anomaly in \( SU(N) \) gauge theories with \( N > 2 \)). The choice of holomorphic theory uniquely specifies the line bundle over \( \mathcal{M} \) whose sections are the holomorphic blocks. For example, \( WZW_4 \) theory of the section 2.1 leads to the line bundle \( \mathcal{L}_S \) where \( S \) is a union of curves, which represent the homology class Poincare dual to the Kähler form \( \omega \).

We shall call these sections instanton theta functions.
5.2. Three, Five, ... - dimensional Chern-Simons theories

In this section we will develop a physical way of counting instanton theta functions.

Recall that we are interested in quantizing the finite-dimensional moduli space \( \mathcal{M} \) of instantons of a given instanton charge. In the holomorphic polarization the wave-functions are the holomorphic sections of the pre-quantization line bundle \( L \) over the space \( \mathcal{M} \). This bundle has a first Chern class coinciding with the symplectic form \( \Omega \). We discussed various forms \( \omega \) which could serve as the symplectic forms, depending on a four-dimensional theory we wish to study. In particular, the avatar of \( WZW_2 \) has relevant \( \Omega \sim \int \omega \wedge \text{Tr} \delta A \wedge \delta A \), whereas the generalized four dimensional WZW theory has \( \Omega \sim \int \text{Tr} F \wedge \delta A \wedge \delta A \).

5.2.1. Supersymmetric quantum mechanics

Given a Kähler manifold one can construct a supersymmetric quantum mechanical model, which would count the number of the holomorphic wave-functions. First of all, naively one doesn’t need any supersymmetry at all, since the dimension of the Hilbert space is given by the path integral of the form:

\[
\dim \mathcal{H}_M \sim \int DpDq e^{-i \int pdq}
\]  

(5.1)

where \( p \) and \( q \) are the Darboux coordinates, and one integrates over the space of loops in \( M \). The measure \( DpDq \) can be defined using a lattice approximation where one replaces the loop by a bunch of points \( t_1 < \ldots < t_N \) and replaces the smooth functions \( p(t), q(t) \) by a collection of points in \( M \): \( q_1, \ldots, q_N, p_{\frac{\pi}{2}}, \ldots, p_{N-\frac{\pi}{2}} \). The reason the momenta \( p_{i+\frac{\pi}{2}} \) have half-integer labels is that the action \( \int pdq \) in the discretized form becomes a sum: \( \sum_i p_{i+\frac{\pi}{2}} (q_{i+1} - q_i) \). But on a curved manifold \( M \) there is no natural way of splitting the coordinates of the point \( x \) into \( p \)'s and \( q \)'s, and what is worse there is no natural reason for the pairing \( p_{i+\frac{\pi}{2}} (q_{i+1} - q_1) \) to be well-defined. So one takes a slightly more complicated route which leads to a nice path integral with more symmetries at the cost of introducing extra fields.

There is a general way of defining a measure in the path integral by introducing the fields of opposite statistics. We introduce the fermionic field \( \psi^\mu \), so that the measure \( Dx D\psi \) is well-defined and invariant under any changes of the coordinates \( x \), provided that they are accompanied by the corresponding change in \( \psi \).
The benefit of the introduced fermions $\psi$ is the supersymmetry:

$$\delta x^\mu = \psi^\mu$$
$$\delta \psi^\mu = \partial_i x^\mu$$

(5.2)

which squares to the translation along the loop $\delta^2 = \partial_t$. In particular it acts as nilpotent operator on the observables, invariant under the rotation of the parameter $t$ and it is possible to define a cohomology space. The symmetry $\delta$ has the following interpretation. Consider the space of parameterized loops $X = LM$. The differential forms on $X$ can be identified with the functionals of $x^\mu(t)$ and $\psi^\mu(t)$, where $\psi^\mu(t)$ corresponds to the differential $dx^\mu(t)$. The circle $S^1$ acts on $X$ by rotations of loops and $\delta$ is the equivariant derivative $d+\iota_V$, with $V$ representing the vector field $\partial_t x^\mu \frac{\delta}{\delta \psi^\mu}$. The action $\int pdq$ now gets replaced by the $\delta$-invariant action

$$\alpha(\phi) = \int_{S^1} dt (\omega_{ij} \psi^i \bar{\psi}^j + \phi \theta_\mu \partial_t f^\mu)$$

(5.3)

Here we introduced a one-form $\theta = \theta_\mu dx^\mu = d^{-1} \omega$, which generally replaces $pdq$. Of course, $\theta$ is defined locally, but the equivariant form $c^\alpha$ would be well-defined even if $\omega$ is not an exact form, but rather closed one with integral periods. We also introduced a number $\phi$ which serves as a normalization constant. It can have only discrete values in order to preserve the integrality of $\alpha$. The number $\phi$ should be thought of degree two generator of $S^1$ equivariant cohomologies.

The partition function is formally independent of any $\delta$-exact terms one could add to the action as long as everything is invariant under the rotations of the circle (and consequently $\delta$ squares to zero). The universal $\delta$-exact regulator, which exists for any Riemannian (not necessarily Kähler or even complex manifold) is the following:

$$\beta(\phi) = \int dt g_{\mu\nu} \left[ \psi^\mu \nabla_t \psi^\nu + \phi \partial_t X^\mu \partial_t X^\nu \right]$$

(5.4)

Where $\nabla_t$ is the pull-back of the Levi-Chivita connection to the circle. If one wouldn’t add the regulator (5.4) and formally integrate out the fermions $\psi$ (since they enter quadratically and ultra-locally), one would get back the integral (5.1). The advantage of having the fermions and symmetry $\delta$ is the possibility to use the localization principle. In this case the fixed points of the group action are the constant loops. Thus, the partition function can be expressed as the integral over the space of constant loops, i.e. $M$ itself. The integrand is given by the ratio of determinants one gets by expanding around the constant loop. It is
well-known that the answer is the index of Dirac operator\textsuperscript{19}. In the case where the action is purely \((5.4)\) it is just the Dirac operator and its index is given by the integral over \(M\) of the \(\hat{A}\)-genus. When \((5.3)\) is added the Dirac operator gets coupled to the connection on a line bundle \(L\), whose curvature is \(\omega\). If one wishes to have \((0, p)\) forms rather then spinors in the target space \(\mathcal{M}\) an extra twist is needed. In fact, one has to couple \(\psi^\mu\) to the square root of the canonical line bundle. The curvature of this bundle is a half of the Ricci tensor (it is represented by \(\frac{1}{2}c_1(\mathcal{M})\)) and the index (partition function) becomes

\[
\int_M Ch(L)\hat{A}(M) = \int_M e^{\omega+\frac{1}{2}c_1(\mathcal{M})}\hat{A}(M) = \int_M e^{\omega}Td(M)
\]

where \(Td\) is the Todd class.

It is clear how to extend this formalism in two respects\textsuperscript{20} (we will need both): the action of a group \(G\) on \(M\) and the quantization of a submanifold \(N\) of \(M\). In the case of our interest the moduli space \(\mathcal{M}\) is a quotient of a submanifold of a Kähler manifold. In the case of a group action the appropriate setting is the equivariant cohomology. To get a submanifold \(N\) one introduces a multiplet of Lagrange multipliers \(H^a\) and their superpartners \(\chi^a\). The number of the \((\chi^a, H^a)\) multiplets equals the codimension of \(N\).

If \(F_a = 0\) are the equations, which define \(N\) (perhaps, locally), then in addition to other terms, the action contains:

\[
\delta \int_{S^1} \chi^a F_a
\]

The supersymmetry \(\delta\) acts on \(\chi, H\) as follows:

\[
\delta \chi^a = H^a \quad \delta H^a = \partial_t \chi^a
\]

If the group \(G\) acts on \(M\) then \(LM\) is acted on by the group \(G = LG \times S^1\), where \(S^1\) acts on \(LG\) by rotations of the loops. The action of the group is incorporated by making \(\delta\) the equivariant derivative with respect to \(G\). One has to introduce a field \(\phi^A(t)\), which takes values in the Lie algebra \(\mathfrak{g}\) of \(G\). Then new derivative is the old one plus a term which acts on \(\psi^\mu\) as \(\phi^A(t) V^\mu_A(x(t))\), and on \(H\) as \(\phi^A(t) T^{ab}_A \chi^b\), where \(V^\mu_A\) is a vector field on \(M\), representing the Lie algebra element \(T_A\), and \(T^{ab}_A\) represents the action of \(G\) on the

\textsuperscript{19} The Dirac operator is the space-time interpretation of \(\delta\)

\textsuperscript{20} For the extended supersymmetry the relevant construction was presented in [62], but here we need to treat the \(N = \frac{1}{2}\) version of the story. Also, we need five dimensional gauge theory.
normal bundle to \( N \). (Of course, for the whole construction to work the submanifold \( N \) must be \( G \)-invariant).

The last piece which changes is the form (5.3). If \( G \) acts symplectically and \( \mu_A \) is the moment map, then (5.3) is replaced by:

\[
\alpha(\phi) = \int_{\mathcal{M}} dt (\omega_{ij} \psi^i \psi^j + \phi \partial_i f^\mu + \phi^A \mu_A) \tag{5.6}
\]

It turns out that if one applies this construction to the space \( M \) being the space of all gauge fields in some vector bundle \( E \) over a two-, or four- complex manifold \( \Sigma \), \( N \) being the space \( \mathcal{M}^{1,1} \) of gauge fields whose curvature has type \((1, 1)\) in the complex structure of \( \Sigma \) (in two dimensions \( N = M \), in four dimensions the equations \( F_\alpha = 0 \) have the form \( F^{2,0} = F^{0,2} = 0 \)) and the group \( G \) being the gauge group, then the quantum mechanical model turns into the

5.2.2. \( N = 2 \) Supersymmetric three-, five- dimensional gauge theory

Consider a rigid minimal \( N = 2 \) supersymmetric gauge theory on a \( d \)-dimensional manifold \( B \). 21 There are two distinct cases of our interest — even-dimensional \( B \) and odd-dimensional \( B \). In the former case for \( d = 2, 4 \) the theory possesses a twisting, which produces a scalar supersymmetry, which is the usual sign of topological theory. This topological theory describes an intersection theory on a appropriate moduli space \( \mathcal{M} \). For \( d = 2 \) it is the moduli space of flat connections, for \( d = 4 \) it is the moduli space of (anti-)selfdual22 connections23 In the case of odd \( d \) the twisting also breaks Lorentz symmetry, and the topological theories can be interpreted as a supersymmetric quantum mechanics on the moduli of \( d - 1 \)-dimensional theory. We will be mainly interested in \( d = 5 \). This theory has real \( \mathfrak{g} \)-valued scalar \( \varphi \), 8 fermions, and 5 components of the gauge fields \( A_M \). After twisting (which uses the decomposition \( 5 \equiv 4 + 1 \) the fermions become:

\[
\psi_{\mu}, \chi^+_{\mu}, \eta
\]

21 They exist for non-abelian groups only in \( d = 2, 3, 4, 5, 6, 7(?) \)

22 it depends on twisting

23 For \( d = 6 \) there is a twist, which breaks Lorentz symmetry, giving rise to the moduli space of holomorphic maps of two-dimensional Riemann surfaces to the moduli of 4-dimensional instantons. After dimensional reduction down to \( d = 4 \) the theory becomes \( N = 4 \) and has three different twists.
where $\psi_\mu$ is four-dimensional vector, $\chi^+$ is four-dimensional self-dual two-form and the susy transformation of our primary interest is:

\begin{align}
\delta A_\mu &= \psi_\mu \\
\delta \psi_\mu &= F_{\mu t} \\
\delta \chi^{+}_{\mu \nu} &= H^{+}_{\mu \nu} \\
\delta H^{+}_{\mu \nu} &= D_t \chi^{+}_{\mu \nu} \\
\delta \varphi &= \eta \\
\delta \eta &= D_t \varphi \\
\end{align} \tag{5.7}

where $t$ is the fifth coordinate, and $\mu$ runs from 1 to 4. In this transformation we recognize the equivariant derivative for the group $G$ of loops in the four-dimensional gauge group, extended by the translations in the $t$-direction. In other words it is the operator $\delta$ in the quantum mechanical problem with infinite-dimensional $M, N, G$. Indeed, take for example the transformation of $\psi_\mu$:

\[ \delta \psi_\mu = F_{\mu t} = D_\mu A_t - \partial_t A_\mu \]

where the first part of the right hand side is the gauge tranformation with the parameter $A_t$ and the second one is the infinitesimal shift $t \rightarrow t + dt$.

The analog of the form (5.6) is the following:

\[ \alpha = \int_{S^1 \times \Sigma} \omega_{\alpha \beta} (CS^3(A)_{\mu \nu} + \frac{1}{2} \text{Tr} \psi_{\mu} \psi_{\nu}) dt \epsilon^{\alpha \beta \mu \nu} d^4x \] \tag{5.8}

with $CS^3$ being the usual Chern-Simons three-form $AdA + \frac{2}{3} A^3$, or

\[ \alpha = \int_{S^1 \times \Sigma} (CS^5(A) + \frac{1}{2} \text{Tr} F \wedge \psi \wedge \psi) dt \] \tag{5.9}

with $CS^5(A) = A(dA)^2 + \frac{3}{2} A^2 dA + \frac{3}{8} A^5$. The reasoning above suggests that the correlator of $e^\alpha$ in the gauge theory on $\Sigma \times S^1$ computes the index of $\tilde{\partial}$ operator, acting in the sections of some line bundle (which depends on the choice of $\alpha$) over the moduli space $M$ of instantons in $\Sigma$. We will be concentrated on the case $G = SU(2)$

Remark. The five-dimensional Chern-Simons theories were studied in [36] as the source of cocycles for four-dimensional theories and in [63] as the generalizations of the three-dimensional topological models. In both cases there were a few drawbacks of the theory.

\[ \text{Right now this is the only case, which seems to be possible to access using string duality. More general gauge groups require knowledge of duals for the higher rank heterotic string compactifications} \]
In the paper [36] there were no principles of choosing the trivial cocycles, while in [63] the structure of the moduli space was found to be quite sick. Both problems are cured by imposing $F^{2,0} = 0$ constraint and choosing the holomorphy as the selection rule.

Loop group gauge theory At this point it is interesting to make the following observation. Consider a topologically twisted $N = 2$ gauge theory on a four dimensional manifold with the gauge group being the central extension of the loop group, extended by the $U(1)$, which rotates the loops. Let $t$ will be the parameter on a loop, $dt - U(1)$-invariant one-form on a circle.

Let us interpret the gauge fields, adjoint scalars and fermions in terms of the usual fields. The gauge field is conviniently packaged as $\nabla_\mu = \partial_\mu + A_\mu$ with $A_\mu = (a_\mu \partial_t + A_\mu; b_\mu)$, where $A_\mu$ has $t$-dependence and is $\mathfrak{g}$-valued, $a_\mu, b_\mu$ are $t$-independent $U(1)$ gauge fields. Similarly the fermions can be decomposed as $\mathcal{W}_\mu = (\psi^a_\mu \partial_t + \psi^b_\mu; \psi^b_\mu)$. Finally, the adjoint scalar has the form: $\Phi = (k (\partial_t + A_t); c)$. It is easy to compute the curvature, supersymmetry transformations and Donaldson observables:

\[ \mathcal{F} = (f_a \partial_t + a \wedge \hat{A} + F_A; f_b + \int dt \text{Tr} A \wedge \hat{A}) \]
\[ \delta A_\mu = \mathcal{W}_\mu \quad \delta \mathcal{W}_\mu = [\nabla_\mu, \Phi] \quad \delta \Phi = 0 \]
\[ \mathcal{O}^2 = \text{Tr}(\Phi \mathcal{F} + \frac{1}{2} \mathcal{W} \wedge \mathcal{W}) = \]
\[ \psi^a \wedge \psi^b + \frac{1}{2} \int dt \text{Tr} \psi \wedge \psi + k \int_C S(A) + k f_b + c f_a + a \wedge \int dt \text{Tr} A_t \hat{A} \]

It is clear that in the gauge $\hat{A}_t = 0$ one recovers (5.7) and (5.8) (up to irrelevant abelian fields $a, b$).

5.2.3. The strategy.

The strategy of using the $N = 2$ supersymmetric gauge theory is the following. We compactify the theory on the circle $S^1$ and expand all the fields in modes along $S^1$. All non-zero modes can be treated perturbatively exactly in one-loop approximation due to supersymmetry. This leads to the expressions we present in the next subsection. They can be also derived directly using the index formula (after all all the statements about exactness of perturbative beta functions [64], [65] in $N = 2$ theories (and perhaps even in $N = 1$) can be traced back to the appropriate statement about cohomologies of the moduli spaces of instantons).
Unfortunately, it seems that this procedure is exact only perturbatively and presently we don’t know how to extend this procedure non-perturbatively. Mathematically the problem seems to be in the compactification of the moduli space of instantons by adding the point-like instantons. The index theorems and statements about the decompositions of characteristic classes are to be modified using the intersection theory on stratified manifolds. This work is still in progress. It would be nice to compare it to the predictions of string duality. Here we sketch a few ideas about the latter.

5.2.4. String theory compactifications

To get a five-dimensional gauge theory with $N = 2$ supersymmetry in a conventional way one has to compactify heterotic string on a $K3 \times S^1$. Then, depending on the choice of $E_8 \times E_8$ bundle (with $c_2 = 24$) over $K3$ one can get various gauge groups in 5d. The model with the minimal number of vector multiplets is obtained by choosing symmetric (12,12) instanton. When further compactified down to 4d one has a model with three vector multiplets: $S, T, U$ (it is rank 4 model of [67][68]) with $S$ being the dilaton and $T, U$ the complex and Kähler moduli of two-torus.

Zero coupling The perturbative moduli space (at zero string coupling) can be described as a Narain moduli space

$$\mathcal{N}_{2,2} = O(2, 2; \mathbb{Z}) \backslash O(2, 2; \mathbb{R}) / O(2) \times O(2)$$

which is just two copies of the modular domains for $T$ and $U$, quotiented by the exchange $T \leftrightarrow U$. It can be identified with a copy of $\mathbb{CP}^2$ using the following standard procedure: $j$-invariant maps the fundamental domain of $SL_2(\mathbb{Z})$ to the $\mathbb{P}^1$. Three points 0, 1, $\infty$ in this $\mathbb{P}^1$ corresponds to the special elliptic curves with $\mathbb{Z}_3, \mathbb{Z}_2$ and $\mathbb{Z}$ symmetries, respectively. Consequently, the metric on the $\mathbb{P}^1$ has orbifold singularities in these points (in particular, the integrand $R\sqrt{g}$ in the Gauss-Bonnet formula has delta-function contributions at these points, making the Euler characteristics equal to $-\frac{1}{12}$, rather than 2). Now we have two copies of $\mathbb{P}^1$ for $T$ and $U$. The last thing is to make a quotient w.r.t. $\mathbb{Z}_2$ which exchanges them. This is done by going to the coefficients of the quadratic equation whose roots are $\tau = \frac{j(T)}{1728}$ and $\nu = \frac{j(U)}{1728}$:

$$Ax^2 + Bxy + Cy^2 = A(x - \tau y)(x - \nu y)$$

---

*Added Sept. 1996.* Now we know how to gain the non-perturbative information. See [66].
Here \((A : B : C)\) are the homogeneous coordinates on \(\mathbb{P}^2\).

The orbifold divisors are the images of the diagonal \(T = U\) and of the points where \(T\) or \(U\) equals to 0, 1, \(\infty\). Thus, we get:

\[
\Theta = \Theta_g \cup \Theta_o \\
\Theta_g = \{B^2 - 4AC = 0\} \\
\Theta_o = \{A = 0\} \cup \{A + B + C = 0\} \cup \{C = 0\}
\]  \hspace{1cm} (5.11)

Two branches \(\Theta_g\) and \(\Theta_o\) have different physical meaning. \(\Theta_g\) corresponds to the \(SU(2)\) gauge symmetry, while the singularities at \(\Theta_o\) generically have no physical meaning and most probably are removed by quantum corrections. The discriminant has three special points, where different branches intersect:

\[
p_\infty = (0 : 0 : 1) \quad p_1 = (1 : -2 : 1) \quad p_0 = (1 : 0 : 0)
\]  \hspace{1cm} (5.12)

where we have written below the gauge groups, which appear at these points as enhancements of \(SU(2)\).

There is by now a substantial evidence that this model has a Type II dual description \cite{67}\cite{68}. Actually, in these papers only affine part of the moduli space \(A \neq 0\) was matched with the Type II description.

Since we are interested in the vicinity of the 5d theory, which is the region, where \(\tau, \nu \rightarrow \infty\), we need a good description of the dual model near this point (actually, a line). The conjecture, that the appropriate description is provided by the 11d supergravity, compactified on a K3 fibration was made in \cite{69}.

**Turning the dilaton on**

At the string coupling turned on the picture changes. It is believed (see \cite{69}) that the perturbative \(SU(2)\) survives quantum corrections in 5d theory. Perturbatively this follows from the structure of anomalies in odd dimensions. This agrees with the investigations of the discriminant locus of dual Calabi-Yau manifold. When further compactified on a circle, the five-dimensional gauge group (which is loops in four-dimensional gauge group) gets Higgsed (by the usual Kaluza-Klein mechanism) and becomes a gauge group of four-dimensional theory.

Taking the radius of the fifth dimension to be finite means stepping aside out of the locus \(\tau = \nu = \infty\).
The moduli space is three-dimensional while for $SU(2)$ theory one needs just two moduli $u$ and $\Lambda_{QCD}$. The resolution of this paradox is simple: since our $SU(2)$ theory comes from five dimensional theory (which contains gravity and $B$-fields in particular) it also must carry Kaluza-Klein $U(1)$'s.

In fact, $(\tau - \nu)^2$ plays the role of four-dimensional $SU(2)$ Higgs field (more precisely, of the order parameter $u$) while $\tau + \nu$ is the scalar component of another $U(1)$ vector multiplet.

The compactification on a two-torus gives four $U(1)$ gauge fields: three of them belong to the vector multiplets $(S, T, U)$ and one is a graviphoton.

It would be nice to get an exact non-perturbative structure of the moduli space we are interested in by isolating the relevant gauge field. We haven't accomplished this yet. The last remark concerns the possible interpretation of the $U(1)$ fields $a, b$ in the loop group gauge theory. Obviously, one of them (which corresponds to the rotations of the circle) is the Kaluza-Klein $U(1)$ gauge field one gets from the five-dimensional metric. It would be interesting to realize the rôle of the dual field.

5.3. Index counting

Suppose the bundle $\mathcal{L}$ is positive. Then the number of its sections can be calculated using the index formula (Riemann-Roch-Grothendieck):

$$\dim H^0(\mathcal{M}, \mathcal{L}) = \int_{\mathcal{M}} e^{c_1(\mathcal{L})} \operatorname{Td}(T\mathcal{M})$$   \hspace{1cm} (5.13)

We need some way of calculating the intersecton pairings between various cohomology classes entering (5.13). First of all, we have to construct these classes.

5.3.1. Constructing the classes

Recall that the moduli space $\mathcal{M}$ can be identified with the (appropriate compactification of) the moduli space of holomorphic bundles $E$ over $\Sigma$ of a given rank (dimension of the fiber). Topologically they are classified by their characteristic classes. Recall, that the holomorphic vector bundle is a vector bundle whose transition functions are holomorphic.

One can try to construct a universal bundle $\mathcal{E}$ over $\mathcal{M} \times \Sigma$ such that its restriction $\mathcal{E}_m$ on $\{m\} \times \Sigma$ for $m \in \mathcal{M}$ gives the holomorphic bundle, which corresponds to $m$. It is non-trivial over $\Sigma$ but it can also be non-trivial over $\mathcal{M}$. (again, the non-triviality is
related to the fact, that one cannot choose the gauge slices for the transition functions uniformly for the whole \( \mathcal{M} \).

Let \( p \) be the projection of the space \( \mathcal{M} \times \Sigma \) to \( \mathcal{M} \). We claim, that the tangent bundle to the space \( \mathcal{M} \) at given point \( m \) can be described as follows:

\[
T_m \mathcal{M} = H^1(p^{-1}(m), \mathcal{E}_m \otimes \mathcal{E}_m^* - 1)
\]  

(5.14)

where 1 is a trivial one dimensional bundle. Since \( \mathcal{A} \otimes \mathcal{A}^* \) is a space of matrices acting in \( \mathcal{A} \), and the scalar matrices act trivially in the adjoint representation, we have to subtract this 1 in counting the automorphisms. The symbol \( H^1 \) is the space of \((0, 1)\)-forms \( a \) with values in \( \mathcal{E}_m \otimes \mathcal{E}_m^* - 1 \), which satisfy the equation \( \bar{\partial}_A a = 0 \) and two such forms are equivalent if the differ by \( \bar{\partial}_A \) of a function.

Indeed, the moduli space \( \mathcal{M} \) coincides with the space of first order differential operators \( \bar{\partial}_A \) acting in \( \mathcal{E} \), such that they square to zero, up to the gauge transformation. The description of \( H^1 \) is the linearization of these conditions. (In general, \( H^1 \) always appears as a tangent space in any moduli problem - this is called a Kodaira-Spencer theory).\(^{25}\) Also, for the moduli space to be a smooth manifold one needs all other cohomology groups to vanish. Therefore, we can use Riemann-Roch-Grothendieck formula\(^{26}\):

\[
Ch(T \mathcal{M}) = -\int_{\Sigma} Td_\Sigma Ch(\mathcal{E} \otimes \mathcal{E}^* - 1)
\]

Let us endow \( \mathcal{E} \) with a unitary connection, such that its component along \( \Sigma \) coincides with the connection \( A, \bar{A} \). Then the curvature \( \mathcal{F} \) of this connection can be decomposed as follows:

\[
\mathcal{F} = \phi + \psi + F
\]

(5.15)

where \( \phi \) is a \((2, 0)\) form, \( \psi \) is \((1, 1)\) and \( F \) is \((0, 2)\) form. Here \((a,b)\) form is a \(a\)-form along \( \mathcal{M} \) and a \(b\)-form along \( \Sigma \) (no to be confused with the Hodge \((p,q)\) decomposition). Then

\[
Ch(\mathcal{E} \otimes \mathcal{E}^* - 1) = \text{Tr} \frac{\mathcal{F}}{2\pi i} \text{Tr} \frac{\mathcal{F}}{2\pi i} - 1
\]

(5.16)

\(^{25}\) As a sheaf: \( T \mathcal{M} = R^1 p_* (\mathcal{E} \otimes \mathcal{E}^* - 1) \)

\(^{26}\) The minus sign comes about because the tangent bundle is given by the first cohomology group which enters the Euler characteristics with the minus sign
Now let us restrict ourselves to the rank 2 case. Then $F$ is just a 2 by 2 traceless matrix-valued 2 form and we can simplify (5.16) as follows:

$$Ch(\mathcal{E} \otimes \mathcal{E}^* - 1) = 2cosh(A) + 1$$

$$A^2 = -\frac{TrF^2}{2\pi^2} \quad (5.17)$$

Recall that $Td_{\Sigma} = 1 + \frac{c_1(\Sigma)}{2} + \frac{c_1(\Sigma)^2 + c_2(\Sigma)}{12}$. Thus,

$$Ch(TM) = \int_{\Sigma} C^{(4)} + C^{(2)}(\frac{c_1(\Sigma)}{2}) + C^{(0)}(\frac{c_1(\Sigma)^2 + c_2(\Sigma)}{12})$$

$$C = 2cosh(A) + 1 \quad (5.18)$$

and superscripts 4, 2, 0 denote the degree of the form on $\Sigma$ which is to be picked out.

In order to convert it to $Td$ we recall that formally there is a relation between the Todd class and the Chern character, which follows from the definitions:

$$logTd(E) = -\frac{c_1(E)}{2} + \int_0^{\infty} \frac{du}{u} \int \frac{dt}{t} \log \left( \frac{u/2t}{\sinh(u/2t)} \right) Ch_t(E)$$

$$Ch_t(E) = \sum_i e^{tx_i} \quad (5.19)$$

and in the contour integral $|t| \gg u$. From the formulae (5.19) and (5.18) we obtain the following answer:

$$logTd(TM) = \int_{\Sigma} W^{(4)} + V^{(2)}(\frac{c_1(\Sigma)}{2}) + U^{(0)}(\frac{c_1(\Sigma)^2 + c_2(\Sigma)}{12})$$

$$W = \frac{A^2}{2} \log \frac{A^2}{\Lambda^2} + 2Li_3(e^{-A}) - \frac{A^3}{6} \sim \frac{A^2}{2} \log \frac{A^2}{\Lambda^2} + Li_3(e^A) + Li_3(e^{-A})$$

$$V = -\frac{A^2}{2} \quad U = \log \left[ \frac{(A/2)}{\sinh(A/2)} \right]^2 \quad (5.20)$$

where $\Lambda$, $\Lambda'$ are some finite numbers (like $e^\pi$) and the sign $\sim$ means the equality up to the terms, linear in $A$ (they don’t contribute to the 4-observable). Quite analogously,

$$logEu(TM) = \int_{\Sigma} P^{(4)} + Q^{(2)}(\frac{c_1(\Sigma)}{2}) + R^{(0)}(\frac{c_1(\Sigma)^2 + c_2(\Sigma)}{12})$$

$$P = \frac{A^2}{2} \log \frac{A^2}{\Lambda^2} \quad Q = \pi i \sqrt{A^2} \quad R = \log \frac{A^2}{\lambda^2}$$

where $\Lambda$ is some finite number, while $\lambda$ is a cut-off. The cut-off, which is required only on the manifolds with $\chi + \sigma \neq 0$ has to do with the infrared divergencies in the quantum theory. It leads to the $\chi, \sigma$-dependent renormalizations and makes the theory more
complicated. In fact, $N = 4$ super-Yang-Mills theory, which was shown in [48] to compute the Euler characteristics of the instanton moduli spaces, cures the problem by introducing extra massless fields. The integral over these extra fields reproduces the leading term (4-observable). The gauge group in the theory is generically broken to an abelian subgroup and the cut-off $\lambda$ which we have introduced comes from the integral over the abelian fields. It can be absorbed into the renormalization of the measure on a curved manifold.

**Virtual abelianization.** Here we would like to explain a little trick which allows to make the computations of the characteristic classes of the tangent bundles of the moduli space in any dimensions fairly easy. It is amusing that it coincides exactly with the physical computation of the beta function in the Coulomb phase. Recall that there is a splitting principle which we used heavily in the transformations from the Chern character to the Todd class and so on. The trick is: assume that the virtual line bundles whose first Chern classes enter the decomposition of the Chern characters are real. So, $\mathcal{E} = \mathcal{L} \oplus \mathcal{L}^{-1}$, $\mathcal{T} \mathcal{M} = H^1(\mathcal{E} \otimes \mathcal{E} - 1) = H^1(\mathcal{L}^2) + H^1(\mathcal{L}^{-2}) + H^1(1)$. Then, $H^1(1)$ gives rise to a trivial bundle over $\mathcal{M}$ so we are only interested in its dimension ($= \text{rank}$). The trivial bundles do not contribute to the Todd class (since $\lim_{\tau \to 0} \frac{\tau - \text{sh}(\tau)}{\text{sh}(\tau)} = 1$). To the contrary the top Chern class (= Euler class) of a bundle $\mathcal{F}$ vanishes if there are trivial bundles in the decomposition $\mathcal{F} = \oplus L_i$. This is why the index computation of the Euler class led to the weird cut-off $\lambda$. It could be that if one replaces the index formula by the real formula for $H^1(1) = h^{0,1}$ then for simply-connected $\Sigma$ the Euler class wouldn’t vanish (at least naively). Anyway, at the moment we are interested in the Todd class and therefore we can discard the $H^1(1)$ issue. So, we have to figure out the contribution of $H^1(\mathcal{L}^\pm 2)$. By the Riemann-Roch-Grothendieck the Chern character of $H^1(\mathcal{L}^\pm 2)$ is given by

$$
Ch(H^1(\mathcal{L}^\pm 2)) = - \int_{\Sigma} e^{\pm 2c_1(\mathcal{L}) + \psi^3 + A/2} T d\Sigma
$$

(5.22)

where $\phi$ has been diagonalized as $\phi = \frac{1}{2} A \sigma_3$, and $\psi^3$ is the projection of $\psi$ onto the Cartan subalgebra determined by $\phi$: $\psi^3 = \frac{1}{2} \text{Tr} \psi \sigma_3$.

Now we are in the position to explain the mnemonic rule. Let us look at the bosonic terms in (5.22):

$$
- \int_{\Sigma} e^{\pm A} P_{\pm}
$$

(5.23)

$$
P_{\pm} = 2c_1^2(\mathcal{L}) \pm c_1(\mathcal{L})c_1(\Sigma) + \frac{1}{12}(c_2^0(\Sigma) + c_2(\Sigma))
$$

One can interpret them as follows: take a (virtual) line bundle $L_{\pm}$ over $\mathcal{M}$ whose first Chern class is given by the zero mode (along $\Sigma$) of $\pm A$. Then the tangent bundle $\mathcal{T} \mathcal{M}$

76
decomposes as a sum of a certain amount of trivial bundles and the bundles $L_{\pm}$. The multiplicity $L_{\pm}$ enters is $\int_{\Sigma} P_{\pm}$. Of course, the precise statement is given by the non-abelian formulas (5.18),(5.20),(5.21). One can get the precise non-abelian formula in the following way. First, one has to compute the characteristic class in question as if the bundle splits indeed and the multiplicities are given by (5.23). One would get an exponential of a polynomial of $c_1(\Sigma), c_2(\Sigma), c_1(L)$ of the form:

$$\int_{\Sigma} p(A)c_1^2(L) + q(A)c_1(\Sigma)c_1(L) + r(A)\left(\frac{c_1(\Sigma)^2 + c_2(\Sigma)}{12}\right)$$

(this is again true up to the $H^1(1)$ issue which would only affect the last term. We shall neglect this.) On the other hand the exact expression will have the form

$$\int_{\Sigma} P^{(4)} + Q^{(2)}\left(\frac{c_1(\Sigma)}{2}\right) + R^{(0)}\left(\frac{c_1(\Sigma)^2 + c_2(\Sigma)}{12}\right)$$

for some gauge invariant functions $P, Q, R$ of $\phi$. It is clear that $p(A), r(A)$ are even functions of $A$, while $q(A)$ is odd. We claim that when restricted to the Cartan subalgebra $P, Q, R$ satisfy:

$$\frac{1}{2}\partial_A^2 P\left(\frac{1}{2} A\sigma_3\right) = p(A)$$

$$\partial_A Q\left(\frac{1}{2} A\sigma_3\right) = q(A)$$

$$R\left(\frac{1}{2} A\sigma_3\right) = r(A)$$

(5.24)

Therefore it is an easy matter to reconstruct $P, Q, R$ once $p, q, r$ are known. This mnemonic rule is really helpful and we will use it below when we shall compute the elliptic genus of $\mathcal{M}$.

5.4. Donaldson theory to the rescue

In two dimensions the theory, which computes the integrals over the moduli space of flat connections is the topological Yang-Mills theory [70], which is a twisted version of $N = 2$ two dimensional super-Yang-Mills theory. The last remark applies equally well to a four-dimensional story. (The last dimension where one has $N = 2$ supersymmetry containing non-abelian gauge fields is $d = 6$ where there is also a moduli space $\mathcal{M}$ of holomorphic bundles and one can identify it with a space of connections, satisfying Donaldson-Uhlenbeck-Yau equations. Physically these equations are just the conditions to preserve supersymmetry). On a Kähler manifolds one can have a twisted version of
$N = 1$ supersymmetric gauge theory and as long as there are no anomalies (since the Kähler twisting is anomalous this requires some matter to be added) the twisted theory is as good as twisted $N = 2$.

Recently the major progress in understanding the physics behind the Donaldson theory has been made by Seiberg and Witten in [71]. It turned out that one can count the Donaldson invariants$^{27}$ in the abelian $N = 2$ gauge theory with the charged hypermultiplet (which represent the monopole field) [72] (on a Kähler manifold one can also use the twist of $N = 1$ theory [73]). In order to use the results of [72] we have to make a crucial assumption which is valid in two dimensional story [70] and wasn’t quite justified in the four dimensional one. Mathematically this assumption is equivalent to the validity of the Künneth decomposition of the cohomology classes of $H^*(\mathcal{M} \times \Sigma)$ together with the assumption on the product structure in the cohomology space.

The reason why we need this is the following. As we presented in the previous section the expression for the Todd class of $\mathcal{M}$ is the exponential of a series, which contains 4 and 2- observables constructed out of the composite operators $u^n \sim (\text{Tr} \phi^2)^n$ for arbitrary $n$. If the assumption is true, then in the $Q$-cohomology the 4-observable of $u^n$ can be represented as:

$$
\int_{\Sigma} (u^n)^{(4)} \sim n(u^{n-1})^{(0)} \int_{\Sigma} u^{(4)} + \frac{n(n-1)}{2} (u^{n-2})^{(0)} \int_{C_{\alpha}} u^{(2)} \int_{C^\alpha} u^{(2)}
$$

(5.25)

where $C_{\alpha}, C^\alpha$ are the dual bases in $H_2(\Sigma)$ and we have assumed the simply-connectness of $\Sigma$.

Assuming the validity of the factorization of the composite observables we write down a an integral formula for the Verlinde numbers:

$$
Z(q; \bar{w}) = \sum_k q^k \chi(\mathcal{M}_k, \mathcal{L}_\bar{w})
$$

(5.26)

Where $\bar{w} \in H^2(\Sigma)$ determines the line bundle over $\Sigma$ and over $\mathcal{M}$ as well. For the four dimensional avatar of 2d WZW theory one has to take $\bar{w} = [\omega]$ - the class of a Kähler form. For simplicity we take $\Sigma = K3$. We use the following normalizations for the correlation functions of the Donaldson observables [73]:

$$
\langle u \rangle = \pm \frac{\pi^2}{4}, \quad \langle \int_{C_{\alpha}} u^{(2)} \int_{C^\alpha} u^{(2)} \rangle = 2 \epsilon \langle u \rangle
$$

(5.27)

$^{27}$ More precisely, the intersection numbers of a restricted class of observables, namely, descendents of $\text{Tr} \phi^2$
Then the answer looks like:

\[
Z(q; \bar{w}) = \frac{1}{2\pi i x^6} \int \frac{t^9 dt}{(t^4 - Q e^{-4G(t)})(1 - e F(t))^{11}} \left( \frac{t/2}{\sinh(t/2)} \right)^4 \exp \frac{(w, w)}{4(1 - e F(t))}
\] (5.28)

where

\[
G(t) = \int_0^1 \log \left( \frac{ty/2}{\sinh(ty/2)} \right) dy = \log(t) + \frac{1}{2t} \left( \text{Li}_2(e^{-t}) - \text{Li}_2(e^t) \right) + \text{const}
\] (5.29)

\[
F(t) = t^3 G'(t), \quad Q = qx^4
\]

Derivation of the formula From the formulae (5.20) and (5.25) we get:

\[
\log Td(M) = 4 \sum_{N=1}^{\infty} (-1)^N \frac{\zeta(2N)}{\pi^{2N}} \left[ -\frac{k}{2N + 1} (u^N)^{(0)} + \frac{N}{2(2N + 1)} (u^{N-1})^{(0)} \int_{C_\alpha} u^{(2)} \int_{C_\alpha} u^{(2)} + (u^N)^{(0)} \right]
\] (5.30)

with \( k \) being the instanton charge and \( x^2 = 4u \). The dimension of the moduli space of instantons of charge \( k \) on \( K3 \) is given by the index theorem: \( \dim \mathcal{M}_k = 8k - 12 \). In order to calculate

\[
\langle e^{(\alpha, u^{(2)}) Td(M)} \rangle
\]

we need two tricks. The first would get rid of the observable \( k \). The problem with it is the following. The 0-observable \( u \) "changes" the instanton number - it has ghost number 4 and the computation in the sector with the instanton number \( l \) and two \( u \)'s inserted is equivalent to the computation in the instanton number \( l - 1 \) without those two \( u \)'s. On the other hand, \( k \) which stands in (5.30) measures the instanton number and therefore the operators inside \( \langle \rangle \) do not commute. It is in fact one of the manifestations of the possible contact terms problem which might spoil the assumptions, leading to (5.25). We shall do the following. Since the total ghost number of the expression inside the brackets must be equal to \( 8l - 12 \) we can compute the correlation function for \( k \) being a \( \alpha \)-number, not related to the instanton charge and then pick out the one with \( k = l \). This is apparently only one
of the possible "normal ordering" prescriptions. To do this for all instanton charges at once we introduce an auxiliary variable \( t \) and rescale all observables according to their ghost charge. Thus, \((u^N)^{(0)}\) gets multiplied by \( t^{2N} \), \( u^{(2)} \) by \( t \) and \( k \) is unchanged (so the power of \( t \) equals half of the ghost number). The next step is to multiply the whole correlation function by \( t^{-4k+6} \), sum over all non-negative \( k \) and take a residue at \( t = 0 \). This gets rid of 4-observables.

The second trick is needed in order to convert the observable \( \int_{C^a} w^{(2)} \int_{C^a} u^{(2)} \) into the expression, linear in \( u^{(2)} \) thereby reducing the computation to the correlation functions of 0 and 2-observables. This was [73] (in agreement with mathematical results of Kronheimer and Mrowka [47]). This is easily done by introducing the auxiliary variable \( \Gamma \) taking values in \( H^2(\Sigma, \mathbb{R}) \) and rewriting the exponent of \( xG'(x) \int_{C^a} u^{(2)} \int_{C^a} u^{(2)} \) as the Gaussian integral:

\[
(xG'(x))^{11} \int_{H^2} D\Gamma e^{\Gamma \int_{C^a} u^{(2)} - \frac{1}{2x^2 C(\omega)} C^{a\beta} \Gamma_a \Gamma_\beta}
\]

(5.31)

where \( C^{a\beta} \) is the inverse of the intersection form, and 11 is \( b_2(K3)/2 \). The intersection form is not positive definite (actually, it is \( 3H \oplus 2E_8 \)), so the integral is defined via the contour rotation prescription.

Assembling the pieces together, evaluating the correlation functions in both vacua (corresponding to the \( \pm \) signs in (5.27)), rescaling \( t \rightarrow tx^4 \) we arrive at (5.28). Obviously, the prescription we have used gives the answer for the general four-manifolds in terms of the numbers of solutions to the monopoles equations [72].

**Comment on compactification of five-dimensional theory** There is one subtlety which was left out in the discussion of the five-dimensional theory and its compactification. When discussing Kaluza-Klein theories one usually distinguishes two situations: one where the non-compact manifold has dimension \( d \geq 4 \) or \( d < 4 \). The distinction comes about because of the role played by the components of the gauge fields along the compactified directions. In the dimension \( d \) the physical scalars have dimension \( \Delta = (d-2)/2 \) as one deduces from the free kinetic term. On the other hand, the gauge field has always dimension 1 due to the gauge transformations. The gauge coupling constant appears as \( \frac{1}{e_D} \) in front of the gauge kinetic energy. Thus \( e_D \) has the dimension \( (4-D)/2 \). Consider the compactification of a gauge theory on a \( D-d \)-dimensional torus. Let \( L_i \) for \( i = 1, \ldots, D-d \) be the circumference of the \( i \)th circle \( S^1 \) of \( T^{D-d} \). The volume \( V_{D-d} \) is the product of all \( L_i \). The \( d \) dimensional theory has the scalars \( \Phi_i \) corresponding to the Wilson loops \( W_i \sim \exp a_i L_i \) around \( S^1 \) (\( a_i \)
is a constant mode of the component $A_i$ of the gauge field). By the counting dimensions argument one easily relates $a_i$ and $\Phi_i$:

$$\Phi_i = \frac{(V_{D-d})^{\frac{1}{2}}}{e_D} a_i$$

The gauge coupling in $d$ dimensions is related to the one in $D$ as

$$e_d = e_D V_{D-d}^{-\frac{1}{2}}$$

The gauge invariance of the $D$-dimensional theory implies that the $d$-dimensional one has to be invariant under the shifts (large gauge transformations):

$$a_i \rightarrow a_i + 2\pi i \frac{n_i}{L_i} \quad n_i \in \mathbb{Z}$$

Now the issue is whether this leads to any constraints on the action written in terms of $\Phi_i$ or not. For our purposes the only relevant cases are $D - d = 1$ or $D - d = 2$. In the former case

$$\Phi = \frac{\sqrt{L}}{e_D} a \rightarrow \frac{\sqrt{L}}{e_D} a + \frac{1}{e_D \sqrt{L}}$$

If $d \geq 4$ then it is $e_d$ which has to be kept finite and therefore the shifts in $\Phi_i$ become infinite as $L \rightarrow 0$ and are not relevant. On the other hand, as $d$ goes below 4 one can keep $e_4$ to be finite and therefore the $e_d$ will go to infinity. This produces no harm in $d = 2$ as the gauge bosons have no degrees of freedom and this "strong coupling" would be artificial. Thus, if $d < 4$, the scalar $\Phi$ starts to live on a curved manifold (a moduli space of flat connections on $T^{4-d}$) and the effective actions must exhibit a periodicity. This explains the origin of the trigonometric (and as we will see, elliptic) functions.

But there must be a loophole in the argument in the higher dimensions, since we were getting the same $\sinh$'s in the expressions for the effective "prepotentials" for precisely the reason - large gauge transformations. It implies, that the scalar $\phi$ (but, perhaps, not its conjugate $\bar{\phi}$) effectively becomes dimensionless in any number of dimensions.

The failure of the dimensional counting argument belongs to the heart of the index theory. In fact, would we follow the argument literally we never get the Todd class and all the complicated expressions, since in terms of the physical field $\phi$ (which enters the untwisted $N = 2$ lagrangian and has the same dimension as $\bar{\phi}$) the formula (5.20) and all the similar ones contain the product $\phi L_5$, where $L_5$ is the circumference of the fifth dimension. By taking the limit $R_5 \rightarrow 0$ one would get rid of all the complications of the
five dimensional theory and would be left with the Donaldson observable \((u^{(2)}, \bar{w})\) together with the action of \(N=2\) twisted Yang-Mills theory.

Mathematically this corresponds to the large \(\bar{w}\) limit, since the volume of the moduli space is the leading term of the formula (5.13).

Physically the limit \(L_5 \rightarrow 0\) was taken when \(\phi\) was kept finite. The characteristic value of \(\phi\) is of the order of \(\Lambda\) - the dynamically generated scale of four-dimensional theory. It is \(\Lambda\) which makes \(\phi\) dimensionless in the ordinary Donaldson theory. In a moment we will see that in the supersymmetric theory, obtained by the compactification the value of \(\Lambda\) has to be correlated with \(L_5\) and in particular one cannot assume \(\Lambda L_5\) to be very small.

Indeed, the procedure of compactification which uses the dimensions and neglects irrelevant operators was simply copied from the bosonic version of the theory. On the other hand, the supersymmetry makes many leading terms in the small \(L_5\) expansion cancel and in particular leads to the vanishing of the most singular terms. Thus, if the observable \((u^{(2)}, \bar{w})\) is not added to the action, the partition function in the non-trivial instanton sector would seem to vanish, as it generically does in the Donaldson theory by the ghost number counting. But the compactified twisted five-dimensional theory calculates the index of the Dirac (or \(\bar{\mathcal{D}}\)) operator on \(\mathcal{M}\) and needs not to vanish.

To get a clearer picture consider the simplest case - compactification of one dimensional theory \((0+1)\). In the case of purely bosonic theory the partition function reduces to the integral over the space of minima of the Hamiltonian and typically is singular in the absence of the potential (we refer to the short time heat kernel expansion). In the presence of fermions and world-line supersymmetry the leading terms of the heat kernel expansion cancel (actually, all \(t\)-dependent terms cancel), leaving one with the index of the supersymmetry generator.

We claim that analogous phenomenon takes place in higher dimensional examples.

Taking the zero modes of the bosonic part of the action and completing it to the supersymmetric action corresponds to the leading term in the heat kernel expansion and leads to the vanishing partition function. One has to perform an accurate expansion around the fields configurations, harmonic along the compact space and take into account the ratio of determinants, which leads to the non-trivial effective action in \(d\) dimensions. It may well happen that the naive dimensions of the fields get changed in an assymetric way (as is twisting).

In conclusion, the consistency check: the variable \(x\) which appeared in the intermediate calculations was actually the product \(\Lambda L_5\). It still makes sense to take various limits for \(x\). Fortunately, up to redefinition of \(q\) and overall normalization, the answer (5.28) is \(x\)-, and therefore \(L_5\)-independent.

82
5.5. Two dimensional Verlinde formula revisited

We conclude by re-deriving the two dimensional formula for the number of holomorphic blocks.

In two dimensions the rôle of the space of instantons is played by the moduli space of flat connections, i.e. the gauge fields with vanishing field strenght, up to gauge transformations. We can check the assertions of the sections 5.1, 5.2. by comparing the result of the application of our method in the two dimensional case, since there the number of holomorphic blocks is known. It was first computed by E. Verlinde in [74], using the sewing and cutting techniques of the two dimensional conformal field theory. Our method involves the intersection theory on the moduli space and thus is more close to the mathematically rigorous proofs of the Verlinde formula. For the proofs and studies of the properties of the formula see [75],[76],[77],[78],[79].

We have discussed an interpretation of this formula in four-dimensional context in the framework of five-dimensional gauge theory. An analogous interpretation of Verlinde formula in two dimensions exists and has to do with Chern-Simons theory, introduced by Witten [80]. In our approach we need a supersymmetric extension of the theory. The idea of applying the supersymmetry for the calculation of Verlinde numbers goes back to Gerasimov [43], although it was noted in [81], that the supersymmetry of [43] is not of the usual kind (in particular, it squares to zero only on special gauge invariant observables. It was sufficient for the purposes of the localization. Another way of deriving the Verlinde formula by means of the direct manipulations with path integral was presented in [82]. Our method also resolves the difficulty of the proper treatment of the endpoint contributions and overall normalizations of [82]). We found a proper way of embedding the Chern-Simons theory into supersymmetric context in the previous section.

We fix a Riemann surface \( \Sigma \) of genus \( g \) and let the gauge group \( G \) be semi-simple Lie group. We also fix a positive integer \( k \), the level.

We are interested in computing the dimension of the space of holomorphic sections of a line bundle \( \mathcal{L} \) over \( \mathcal{M} \), whose first Chern class equals \( k \) times the Kähler form on \( \mathcal{M} \). By the previous arguments it is given by the index theorem and so our task is to compute the intersection numbers of the classes \( Td(\mathcal{M}) \) and \( \exp c_1(\mathcal{L}) \). We shall reduce this problem to the problem of calculations of the correlation function in the two dimensional Yang-Mills theory of the 2-observables and 0-observables:

\[
\langle \mathcal{O}^{(2)}_T \mathcal{O}^{(0)}_P \rangle_{\text{YM}_2} \tag{5.32}
\]

83
where, by the general scheme, we have:

\[ T = - \frac{(k + h^\vee)}{8\pi^2} \text{Tr}(\phi^2) \]

\[ P = \prod_{\alpha > 0} \left[ \frac{\langle \phi, \alpha \rangle / 2(2\pi i)}{\sinh(\langle \phi, \alpha \rangle / 2(2\pi i))} \right]^{2(g-1)} \quad (5.33) \]

where \( h^\vee \) is the dual Coxeter number of \( G \). Now recall, that in two dimensional Yang-Mills theory the correlation function of the 0-observable in the presence of 2-observable of the type (5.33), i.e. the descendent of the quadratic casimir is easily computed. One has to rescale the variable \( \phi \) as well as \( \psi \) in order for the 2-observable part of the action to have a canonical form. This yields a factor

\[(k + h^\vee)^{\text{dim}G(g-1)}\]

in front of the answer. Then the evaluation proceeds as follows: one replaces \( \frac{\phi}{2\pi i} \) everywhere by the highest weight \( \hat{h} \) of irreducible representation \( \rho \) of \( G \), shifted by the half of the sum of positive roots \( \delta \), divides by the \( 2g - 2 \)nd power of the dimension of the representation \( \rho \), multiplies by \( e^{-\epsilon \langle \hat{h} + \delta, \hat{h} + \delta \rangle} \), takes the sum over all \( \rho \) and takes the limit \( \epsilon \to 0 \) (second reference in [70]). In formulas this looks as follows:

\[ \left\langle e^{\frac{h + h^\vee}{4\pi^2} \int_{\Sigma} \text{Tr}(i\phi F + \frac{i}{2} \psi \wedge \psi)} P(\phi) \right\rangle = \]

\[(k + h^\vee)^{\text{dim}G(g-1)} \left\langle e^{\frac{1}{4\pi^2} \int_{\Sigma} \text{Tr}(i\phi F + \frac{i}{2} \psi \wedge \psi)} P\left(\frac{\phi}{(k + h^\vee)}\right) \right\rangle = V_{\text{pert}}(\epsilon)_G,k,g \]

\[ V(\epsilon)_G,k,g = (k + h^\vee)^{\text{dim}G(g-1)} \]

\[ \sum_{\rho} \left[ \frac{1}{\text{dim}(\rho)} \prod_{\alpha > 0} \frac{\langle \hat{h} + \rho, \alpha \rangle}{2(k + h^\vee)} \sinh(\pi \frac{\langle \hat{h} + \rho, \alpha \rangle}{k + h^\vee}) \right]^{2(g-1)} e^{-\epsilon \langle \hat{h} + \rho, \hat{h} + \delta \rangle} = \]

\[ \left[ \frac{k + h^\vee}{2} \right]^{\text{dim}G(g-1)} \sum_{\rho} \frac{e^{-\epsilon \langle \hat{h} + \rho, \hat{h} + \delta \rangle}}{\text{dim}(\rho) \prod_{\alpha > 0} \sinh(\pi \frac{\langle \hat{h} + \rho, \alpha \rangle}{k + h^\vee})}^{2(g-1)} \]

For \( G = SU(2) \) the last formula has a form:

\[ V(\epsilon)_{SU(2),k,g} = \left(\frac{k + 2}{2}\right)^{(g-1)} \sum_{n=1}^{\infty} \sinh^{2-2g}(\frac{\pi n}{k + 2}) e^{-\epsilon \frac{k + 2}{2} \pi n^2} \to \]

\[ \left[ \frac{k + 2}{2} \right]^{(g-1)} \sum_{n=1}^{\infty} \sinh^{2-2g}(\frac{\pi n}{k + 2}) \quad \text{pol} \quad \]

\[ 84 \]
where the brackets $[.]_{\text{pol}}$ is the notation for the polynomial in $k + 2$ part.

On the other hand the Verlinde formula for $SU(2)$ has the form:

$$\dim H^0(M, \mathcal{L}^k) = \sum_{n=1}^{k+1} \left[ \frac{k + 2}{2 \sin^2 \left( \frac{\pi n}{k+2} \right)} \right]^{g-1}$$  \hspace{1cm} (5.36)

We claim that these formulas coincide when $k + 2$ is replaced by $i(k + 2)$. Expansion in $\frac{1}{k+2}$. It is instructive to perform a $\frac{1}{k+2}$ expansion of (5.36) to compare the answer with

$$\kappa^p \sum_{n=1}^{\infty} \frac{1}{\sinh^2(\frac{\pi n}{\kappa})}$$  \hspace{1cm} (5.37)

for $\kappa = k + 2$, $p = g - 1$. We can rewrite (5.36) as follows:

$$(5.36) = (-2\kappa)^p \kappa \int_{\Gamma} \frac{t^{2\kappa} dt}{2\pi i t (t - t^{-1})^{2p}(t^{2\kappa} - 1)} =$$

$$\left( \frac{\kappa}{2} \right)^p \kappa \int_{\Gamma} \frac{x^{\kappa+p} dx}{2\pi i x (x - 1)^{2p}(x^{\kappa} - 1)} = \text{Res}_{x=1} =$$

$$(-)^{p-1} \kappa^{p+1} 2^p \sum_{n=1}^{\infty} \left( \frac{n\kappa + p - 1}{2p - 1} \right)$$  \hspace{1cm} (5.38)

where it is understood that the binomial coefficients are expanded as polynomials in $n$ of degree $2p - 1$ and the sums $\sum_{n=1}^{\infty} n^l$ are defined as the values of $\zeta$-function at $-l$. This prescription is equivalent to the $e^{-\epsilon n^2}$ regularization. Thus:

$$\sum_{n=1}^{\infty} n^{2l} \to 0$$

$$\sum_{n=1}^{\infty} n^{2l-1} = \zeta(1 - 2l) = \frac{\zeta(2l)}{(-4\pi^2)^l}$$

We also have:

$$\sum_{n=1}^{\infty} \left( \frac{n\kappa + p - 1}{2p - 1} \right) = \frac{\kappa^{2p-1}}{(2p-1)!} \sum_{n=1}^{\infty} \left( \frac{n + p - 1}{\kappa} \right) \ldots \left( n - \frac{p}{\kappa} \right) =$$

$$= \frac{\kappa^{2p-1}}{(2p-1)!} \sum_{l=0}^{p} \frac{c_{2l}}{\kappa^{2l}} \zeta(1 - 2p + 2l)$$  \hspace{1cm} (5.39)

85
where \( a_I \) is given by the sum over all subsets \( I \) of \( \{-p, \ldots, p-1\} \) of cardinality \( l \) of the products of all elements of \( I \):

\[
c_l = \sum_{I \subset \{-p, \ldots, p-1\}, \# I = l} \prod_{x \in I} x
\]

In particular, \( c_0 = 1, c_1 = -1, c_2 = -\frac{1}{6}p(p-1)(2p-1) \), etc. Combining pieces together we get the following expansion of (5.36):

\[
k^{3p-2p+1}(\kappa) \sum_{l=0}^{p-1} \frac{(2p-2l-1)!}{(2p-1)!} \kappa^{2l} \zeta(2p-2l) \frac{1}{(2\pi i\kappa)^{2p-2l}} =
\]

\[
-\frac{2^{1-p}}{\pi^{2p}} \kappa^{3p} \zeta(2p) \left( -\frac{1}{6}p(p-1)(2p-1)\kappa^{3p-2} \zeta(2p-2) \frac{1}{(2p-1)(2p-2)} \right) + \ldots
\]

\[
= \frac{-2^{1-p}}{\pi^{2p}} \left( \kappa^{3p} \zeta(2p) + \frac{p\pi^2}{3} \kappa^{3p-2} \zeta(2p-2) + \ldots \right)
\]

(5.40)

At the same time, one can easily expand the sum of the inverse sinh's. In particular, the first subleading term comes out with an opposite sign (it seems plausible that the difference between (5.37) and (5.36) is precisely in the sign of the terms \( \frac{1}{\kappa^{4k-2}} \)):

\[
k^p \sum_{n=1}^{\infty} \frac{1}{\sinh^{2p}(\pi n/\kappa)} \sim \kappa^{3p} \left[ \zeta(2p) - \frac{\pi^2}{3\kappa^2} p \zeta(2p-2) + \ldots \right]
\]

The reason for the discrepancy we found is that the form on the moduli space \( \mathcal{M} \), which corresponds to the observable \( \frac{2i}{4\pi^2} \int_\Sigma \text{Tr}(i\dot{\phi}F + \frac{1}{2}\psi \wedge \psi) \) differs from the canonically normalized form \( \omega \) by a factor of \( i = \sqrt{-1} \). This goes back to the relation between the topological Yang-Mills theory (which computes the integrals over the moduli space \( \mathcal{M} \) and is the twisted \( N = 2 \) super-Yang-Mills theory) and the physical one, where only half of the fermions and only part of the scalars is present at the cost of non-perturbative (in \( \epsilon \)) corrections to the topological correlation functions. One cannot set the coefficient in front of \( \int_\Sigma \text{Tr}(i\dot{\phi}F + \frac{1}{2}\psi \wedge \psi) \) to be an arbitrary number, simply because it is only with the imaginary coefficient this term provides the localization on the flat connections in the absence of the multiplet of the cohomological theory. On the other hand, since the answer depends polynomially on the coefficient in front of the symplectic form it can be continued to the region of our interest.
5.6. Elliptic genera and generalizations

One can compute the elliptic genus of $\mathcal{M}$ and the one in the presence of the line bundle $\mathcal{L}$. We want to digress and to discuss a six-dimensional minimal $N = 1$ gauge theory. Consider the manifold $\Sigma \times C$ with $C$ being two dimensional Riemann surface. We denote the complex coordinates on $C$ as $z, \bar{z}$. The left spinors on $C$ will be denoted as $\psi_+$, and the right ones as $\psi_-$. One can make a twist along $\Sigma$ (as opposite to what has been proposed in [83]) and get a theory, whose fermionic content looks as follows:

$$
\psi_{\mu+} \quad \eta_- \quad \chi^+_{\mu-}
$$

where $\mu$ is the index, tangent to $\Sigma$. The theory has a supercharge $Q_+$, which is a scalar along $\Sigma$ and positive (left) spinor along $C$:

$$
Q_+ A_\mu = \psi_{\mu+} \quad Q_+ \psi_{\mu+} = F_{\mu z}
$$
$$
Q_+ \chi^+_{\mu-} = F^+_{\mu \nu} \quad Q_+ A_z = 0
$$
$$
Q_+ A_\bar{z} = \eta_- \quad Q_+ \eta_- = F_{\bar{z} \bar{z}}
$$

(5.42)

If the manifold $\Sigma$ is Kähler then one can decompose the charge $Q_+$ as a sum of two: $\delta'_+$ and $\delta''_+$, which are the generalizations of the charges one has on a Kähler manifold when $N = 2$ supersymmetry is twisted. Now, if $C$ is a two-torus, then $Q_+$ becomes a global supersymmetry and as the action is essentially $Q_+$ - exact one can evaluate the partition function in the limit where $\Sigma$ is very small. In that limit the six-dimensional model becomes a sigma model with a target space being the moduli space of instantons on $\Sigma$ (analogously to [84]). For the smooth instanton moduli space the fields $\chi, \eta$ have no zero modes, therefore the sigma model will have only left-moving and no right-moving fermions. Its partition function is equal to

$$
\text{Tr}_{H(-)} F \rho^{L_0} \rho^{\bar{L}_0}
$$

The sigma model would be conformally invariant if $c_1(\mathcal{M}) = 0$. The latter is equivalent to $c_1(\Sigma) = 0$. Even if it is not the case, then the Hamiltonian in the quantum mechanical problem splits as a sum of the left-moving $L_0$ and the right-moving $\bar{L}_0$. As $Q^2_+ = L_0 = \partial_\Sigma$ the answer is $q$ independent and can be computed on the zero modes of $L_0$. It is given by [85]:

$$
G(q) = q^{-\frac{d_3}{2}} \int_{\mathcal{M}} \hat{A}(\mathcal{M}) Ch(\otimes_{\Sigma} S_q T)
$$

87
where \( d \) is the dimension of \( \mathcal{M} \), \( T \) is the tangent bundle and all other notations are explained in appendix. In the sequel we change the notation \( q \leftrightarrow q \). As before the computation is essentially the converting the "abelian" answer into the gauge and supersymmetry invariant one. It is sufficient to present the change in the function \( p \) only:

\[
\Delta p(A) \sim \sum_{n=1}^{\infty} -log(1 - q^n e^A) - log(1 - q^n e^{-A})
\]

From this one deduces that the "prepotential" \( W \) gets extra trilogarithms:

\[
W(A^2, q) = \frac{1}{2} A^2 \log \frac{A^2}{A^2} + \sum_{n=0}^{\infty} \text{Li}_3(q^n e^A) + \text{Li}_3(q^n e^{-A}), \quad (5.43)
\]

\( V \) is not changed and

\[
U = \log \left[ A^2 \prod_{n=0}^{\infty} \frac{1}{(1 - q^n e^A)(1 - q^n e^{-A})} \right]
\]

5.6.1. Superstring Prepotentials, Six-dimensional Theory, Quantum Affine Algebras and Hopes for the Future

The interpretation of the formulas like (5.43) in four and two dimensions is as follows. The sigma model with the target space \( \mathcal{M} \) appears in the limit, where the volume of the manifold \( \Sigma \) is small. One can take the opposite limit, where the metric on \( \Sigma \) is scaled up. In this limit one gets a sigma model whose worldsheet is \( \Sigma \) and target space is the moduli space of flat connections on a two-torus. Now what are the supersymmetries? We have twisted the original \( \mathcal{N} = 1 \) susy along \( \Sigma \) and because the torus has the covariantly constant spinors the supercharges \( Q_+ \) (and \( Q_+ \) for Kähler \( \Sigma \)) become scalars on \( \Sigma \). So we get a topological field theory. In two dimensional case it is easy to figure out that it is a sigma model of type \( B \), since the crucial part of the supersymmetry transformations

\[
Q \psi \sim d_A \phi
\]

implies that the theory localizes onto the constant maps to the moduli space of flat connections on two-torus (\( \phi \) up to the gauge transformations being the local holomorphic coordinate). As it is well-known the correlation functions in the \( B \)-model are independent of the Kähler class of the target space but they depend on the complex structure. The complex structure is encoded in \( \tau \), thus we expect the answer to be the function in \( q = e^{2\pi i \tau} \).
Usually, the correlation functions in the $B$-model are the sections of some line bundles over the moduli space of complex structures. Would the moduli space of flat connections on torus be the Calabi-Yau manifold the statement will apply and we would conclude that the elliptic genus of $\mathcal{M}$ is a modular form. But, as it is known, it is necessary for $\mathcal{M}$ to be have a vanishing first Pontryagin class for the elliptic genus to be modular.

On the other hand, the moduli space of flat connections on a torus has the structure of an orbifold and naively has non-vanishing first Chern class. This would lead to the anomaly in the sigma model as on the compact target space one needs a holomorphic top form to make the correlation functions well-defined [86].

Nevertheless, it seems that the gauge coupling generates the effective superpotential (in four dimensions - prepotential), which ”doesn’t allow” the worldsheet to get into the region where the top form has poles. Let us illustrate the difficulty on the example of two dimensional $\Sigma$. The calculation we did goes through quite analogously and we arrive at the integrand of the form:

$$P = q^{1/8} \frac{1}{\vartheta_1^2(a; \tau) \vartheta_2'(0; \tau)}$$

where $\text{Tr} \psi^2 = -2\pi^2 a^2$. One can write everything in terms of the abelian fields. In particular,

$$\phi = \pi i \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$$

We can change the coefficient in front of the 2-observable by adding a symplectic form. At the level of four-dimensional theory on $\Sigma \times T^2$ this would correspond to the term like

$$A_z F_{w\bar{w}} + \text{fermions}$$

which one can make invariant under (small) gauge transformations by adding a term, which vanishes on the localization locus. We get:

$$\text{Tr}(A_w \partial_z A_{\bar{w}} + A_z F_{w\bar{w}}) + \text{fermions}$$

which is a supersymmetric-invariant extension of the component $zw\bar{w}$ of a Chern-Simons three-form. It is not invariant under the large gauge transformations. This is in accordance to the fact that if one shifts the value of the zero mode of $a$ by a multiple of $\tau$ the exponential of two-observable gets shifted by a power of $q$ (as the integral of $\text{Tr}(F\sigma_3)$ over $\Sigma$ is an integer multiple of $2\pi i$. What is much worse, is that the theta-functions in the numerator
of (5.44) multiply the whole integrand by \( \exp(4\pi i(g-1)a) \). This cannot be compensated by a shift in the instanton number. So, in order to heal the lack of gauge invariance one has to modify the model. One possibility is to add extra fermions, so that the model becomes more balanced and reproduces another sort of elliptic genus, namely \( F(q) \) (see Appendix). There we would have, instead of (5.44):

\[
(O_P^{(0)})^{g-1} \exp \frac{2}{4\pi^2} \int_\Sigma \text{Tr}(i\phi^0 F + \frac{1}{2} \psi \wedge \psi) \\
P = q^{3/8} \frac{\partial_1(a + \frac{1+i\tau}{2}; \tau) \partial_1(a - \frac{1+i\tau}{2}; \tau) \partial_1 \left( \frac{1+i\tau}{2}; \tau \right)}{\partial_1^2(a; \tau) \partial_1'(0; \tau)}
\]

(5.45)

Quite interesting is the fact that except for the 2-observable, the rest is completely gauge invariant (in particular, double-periodic in \( a \)). The further exploration of these properties and their relevance to the studies of moduli spaces will be discussed elsewhere.

It is amusing to note the similarity of the formulas (5.20), (5.43) and the perturbative prepotential calculations of [87]. One could hope to extend this similarity into the deeper relation and using the string duality to get the full non-perturbative answer. It would be nice if the assumptions, which led to (5.28), get justified. One of the reasons the connection might exist is that the elliptic genus of \( \mathcal{M} \) could appear as a partition function in the twisted minimal \( N = 1 \) supersymmetric gauge theory on \( \Sigma \times T^2 \). The interest to the elliptic genera computations is also motivated by the possible light it could shed on the geometric origin of quantum affine algebras. To this end one has to study \( N = 1 \) SYM in four dimensions on \( \Sigma \times T^2 \) for two dimensional \( \Sigma \).
5.7. Appendix. Twisted $N = 2$ Theory, Localization, Elliptic Genera and Theta-Functions

5.7.1. Remarks on equivariant cohomology and localization

Suppose one is given a Kähler manifold $X$ with a holomorphic action of some complex group $G$. One can construct the usual equivariant complex $\Omega^*_G(X)$ for the maximal compact subgroup $G$ of $G$. Recall, that the elements of the complex (in the Cartan model) are the $G$-equivariant functions on the Lie algebra $\mathfrak{g}$ of $G$ with values in differential forms on $X$.

The action of $G$ will be temporarily assumed to have an equivariant moment map $\mu : X \to \mathfrak{g}^*$. It implies, that to any given element $\phi \in \mathfrak{g}$ corresponds a Hamiltonian vector field $V_h(\phi)$, s.t. $\iota_{V_h}\omega = d < \mu, \phi >$ where $\omega$ is a Kähler form. There is also another vector field $V_g(\phi)$ (which is also present in almost Kähler situation), which generates the gradient flow of the function $< \mu, \phi >$. The holomorphic action of the complex group $G_\mathbb{C}$ is generated by the vector fields

$$v(\phi) = V_h(\phi_1) + iV_g(\phi_2)$$

with $\phi = \phi_1 + i\phi_2 \in \mathfrak{g}_\mathbb{C}$.

Suppose we are given an equivariant form $\alpha$. Consider the following family of forms:

$$\alpha_t(\phi) = (\exp(it\phi))^*\alpha(\phi)$$

where now $\phi \in \mathfrak{g}, i\phi \in \mathfrak{g}_\mathbb{C}$. It is clear, that $\exp(it\phi)$ is generated by the vector field $V_g(\phi)$. We claim that all $\alpha_t$ are equivariant and moreover, $\alpha_t$ is equivariantly closed iff $\alpha$ is. The proof is straightforward. First of all, equivariance means that for $g \in G$ $\alpha(\phi^g) = g^*\alpha(\phi)$ and this obviously holds for $\alpha_t$ since $\exp(it\phi^g) = g^{-1}\exp(it\phi)g$. To check the second property, one makes use of the fact, that the action of $G_\mathbb{C}$ is holomorphic, therefore $[V_h(\phi), V_g(\phi)] = 0$. Hence,

$$(\exp(it\phi))^*V_h(\phi) = V_h(\phi)$$

The form $\alpha_t$ is cohomologous to $\alpha$ for all values of $t$. Actually, $\alpha_t = \exp(-itD_tV_g(\phi))\alpha_0$

Now let us consider some examples.

Yang-Mills theory, $X = \mathcal{A}$ is the space of connections (gauge fields) in the principal $H$-bundle $\mathcal{E}$ over a complex manifold $M$. 

91
\( G \) is a gauge group, \( G_{\mathbb{C}} \) is a complexified gauge group. The holomorphic action of \( G_{\mathbb{C}} \) is defined by the usual gauge action on the \((1,0)\) components of the gauge field.

Take
\[
\alpha(\phi) = \int_C \text{Tr}(\phi F + \frac{1}{2} \psi \wedge \psi)
\]
for \( C \) - two-cycle in \( M \). The claim is:
\[
\int_0^1 dt \alpha_t = S_{GWZW}(h, A, \vec{A}) + \int_C \text{Tr}(\psi T(h) \bar{\psi})
\]
where \( h = \exp(2i\phi) \), \( T(h) = \frac{1 - A d(h)}{ad(2i\phi)} \) and \( S_{GWZW} \) is a gauged \( G/G \) two dimensional WZW theory action.

Index theorems. \( X \) - space of parameterized loops in some Kähler manifold \( M \). \( X \) has a complex structure, which can be described by identifying it with the space of holomorphic maps \( f : \mathbb{C}^* \to M \).

\( X \) is acted on by \( \mathbb{C}^* \subset \mathbb{G}_{\mathbb{C}} \). Consider first the action of \( \mathbb{C}^* \subset \mathbb{G}_{\mathbb{C}} \):
\[
\lambda \in \mathbb{C}^* : (f^i(z), f^\bar{i}(\bar{z})) \to (f^i(\lambda z), f^\bar{i}(\lambda \bar{z}))
\]
(the generators of this action are, of course, \( L_0 \) and \( \bar{L}_0 \)). The action of \( U(1) \) preserves the one-form \( dt = dz/z \) on the circle \( S^1 \). Consider the following \( U(1) \) - equivariant form:
\[
\alpha(\phi) = \int_{S^1} dt (\omega \bar{\partial} \bar{\psi} \bar{\partial} \bar{\psi} + \phi \partial \mu \partial_t f^\mu)
\]
(5.47)

where the Kähler form \( \omega \) is (locally) \( d\theta \) (as usual, the existence of \( \alpha \) can be relaxed to the existence of \( \exp(\alpha) \) which allows \( \omega \) to be not exact, but rather integral). Then:
\[
\alpha_\tau = \int_{S^1} dt (\omega \bar{\partial} \bar{\psi} (f^k(t + i\tau \phi), f^\bar{k}(t - i\tau \phi)) \bar{\psi}(t + i\tau \phi) \bar{\psi}(t - i\tau \phi) +
\]
\[
+ \partial_{\tau} K(f^k(t + i\tau \phi), f^\bar{k}(t - i\tau \phi))
\]
(5.48)

where \( K \) - is a Kähler potential. One could rewrite (5.48) with the help of superintegrals: introduce \((1|2)\) -dimensional worldline, with the coordinates \( t, \theta, \bar{\theta} \) and consider the following ”chiral” superfields:
\[
\hat{X} = X(\hat{t}) + \theta \psi(\hat{t}), \hat{\bar{X}} = \bar{X}(\hat{t}) + \bar{\theta} \bar{\psi}(\hat{t})
\]

where \( \hat{t} = t + i\tau \phi + \theta \bar{\theta}, \hat{\bar{t}} = t - i\tau \phi - \theta \bar{\theta} \). Then
\[
\alpha_\tau = \int dt d\theta d\bar{\theta} K(\hat{X}, \hat{\bar{X}})
\]

92
The first variation of $\alpha_\tau$ w.r.t $\tau$ exists for arbitrary Riemannian manifold:

$$\beta(\phi) = \frac{d}{d\tau}|_{\tau=0}\alpha_\tau = \phi \int dt g_{\mu\nu}(\psi^\mu \nabla_\nu \psi^\nu + \phi \partial_\mu X^\mu \partial_\nu X^\nu)$$

It is used in the path integral proofs of Atiyah-Singer index theorem for Dirac operator.

If one uses the form $\alpha_\tau$ instead of $\beta$ one gets an index of Dirac operator, coupled to the line bundle, whose curvature is the Kahler form $\omega$.

Recall, that on a Kähler manifold one can identify the spinor bundles with the bundles of $(0, q) -$ forms [32], and the Dirac operator with the operator $\bar{\partial} \oplus \partial$. This involves "twisting" by the square root of the canonical bundle. Therefore, left-handed spinors coupled to $K^{1/2}$ are the same as the $(0, 2n + 1)$ forms and thereby the index theorem for Dirac operator must reduce to the one for the $\bar{\partial}$ operator, coupled to some line bundle.

Indeed, it is the case. Atiyah-Singer theorem gives the formula:

$$\text{Ind}(\bar{\partial}) = \int_M Ch(\mathcal{L}) \tilde{A}(M) = \int_M \exp(c_1(\mathcal{L}) + \frac{1}{2} c_1(M)) Td(M)$$

where the right hand side is the Riemann-Roch formula.

### 5.7.2. Twisted $N = 2$

The restriction to the space $A^{1,1}$ is most easily adapted by introducing the Lagrange multipliers and ghosts, with appropriate BRST symmetry. For the general introduction to the equivariant cohomology and related subjects one may consult [88].

Let us discuss the BRST symmetry. Consider the topological Yang-Mills theory in 4 dimensions. The field content of this theory can be organized in the multiplets:

$$\begin{array}{cccc}
-2 & -1 & 0 & 1 \\
A & \psi & \phi \\
\bar{\phi} & \eta & \chi & H
\end{array}$$

where $A$ is a gauge field, $\psi$ - one-form, $\eta, \phi, \bar{\phi}$ - scalars, $\chi, H$ - self-dual two forms - all in the adjoint representation. Also, $A, \phi, \bar{\phi}, H$ are the bosonic fields, while $\psi, \chi, \eta$ - fermionic ones, and upstairs the ghost numbers of the fields are listed. The symmetry in question acts as follows $\delta \Psi = -i\{Q, \Psi\}$:

$$\begin{align*}
\delta A &= \psi, & \delta \psi &= d_A \phi, & \delta \phi &= 0, \\
\delta \chi &= H, & \delta H &= [\chi, \phi], \\
\delta \bar{\phi} &= \eta, & \delta \eta &= [\bar{\phi}, \phi].
\end{align*}$$

93
Mathematically this symmetry is the equivariant derivative acting in the space of equivariant forms on the space of gauge fields. It is nilpotent on the gauge-invariant functionals. Physically it is the field content of twisted $N = 2$ Yang-Mills theory, $H$ being the auxiliary field.

The action of the theory is

$$S = \frac{i\tau}{4\pi} \int \text{Tr}(F \wedge F) + \{Q, R\}$$

$$R = \int \text{Tr}(\chi F^+ - e^2 HF^+ + d_A \phi \star \psi)$$

(5.49)

where $F^+$ denoted the self-dual part of the curvature $F$, $e^2$ is a coupling constant and $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$.

It is clear how to generalize the field content, the action and the symmetry to the five-dimensional case. One possibility is to declare that the gauge symmetry of the five-dimensional theory is the group of loops to the 4d gauge group, extended by the circle, which rotates these loops. In that case the scalar $\phi$, which was a generator of the gauge transformations in 4d SYM gets promoted to a gauge field along the circle. More concretely, let $t$ be the fifth coordinate (it runs from 0 to $2\pi R$. We replace $\phi$ by the operator:

$$\partial_t + A_t$$

in all the formulae for the action of $\delta$. The gauge invariance treats $\phi$ and $\phi$ differently: $\phi$ transforms as a gauge field, while $\bar{\phi}$ is still a scalar in the adjoint representation.

As in the usual topological theories one has to look at the cohomology of the operator $\delta$. They provide a set of observables one can insert into the path integral. In the usual Donaldson (twisted $N = 2$ SYM) theory the correlation functions of the observables turn out to be equal to the integrals over the moduli space $\mathcal{M}$ of instantons of some differential forms. Usually one interprets these integrals as intersection numbers of some submanifolds (actually, homology cycles) of $\mathcal{M}$. We will interpret them as symplectic volumes of some moduli spaces (we will explain this later). Then the correlation functions of the $\delta$-closed observables in 5d theory correspond to the dimensions of the spaces, one can get by quantizing these moduli spaces with the symplectic forms.

In Donaldson theory, the natural operators to start with are the invariant polynomials of the variable $\phi$ evaluated at some point $x \in \Sigma$:

$$\mathcal{O}_P(x) = P(\phi(x)) = \sum_n p_n \text{Tr}(\phi^n(x))$$

94
They are annihilated by $\delta$ and are gauge invariant. The correlation function of $\delta$-closed observables with inserted $O_P^0(x)$ is $x$ independent, since the derivative of $O_P^0(x)$ is $\delta$ of something. In fact, one build a chain of equations (called the descend equations):

\[
\begin{align*}
    dO_P^0(x) &= \delta S_P^1 \\
    dS_P^1 &= S_P^2 \\
    \ldots
\end{align*}
\]

(5.50)

Then, for a closed surface $C_i$ in $\Sigma$ of dimension $i$ ($i$-cycle) the observable $\int_{C_i} S_P^i$ is $\delta$-closed and depends up to $\delta$-exact terms only on the homology class of $C_i$. Let us define $O_{P,C}^i = \int_{C_i} S_P^i$. It is convenient to package all $S_P^i$'s at once as follows:

\[ P(\phi + \psi + F) = \sum_i S_P^i \]

(5.51)

In particular, for $P = \frac{1}{2} \text{Tr}(\phi^2)$ one has

\[
\begin{align*}
    S_P^1 &= \text{Tr}(\phi \psi), \\
    S_P^2 &= \text{Tr}(\phi F + \frac{1}{2} \psi \land \psi), \\
    S_P^3 &= \text{Tr}(\psi \land F), \\
    S_P^4 &= \text{Tr}(F \land F)
\end{align*}
\]

(5.52)

At this point it is useful to look at the observables, which would replace Donaldson observables in $5d$ theory. It is clear that taking the trace should be accompanied by the integration over $dt$ in order to make things invariant under the time translations which are the part of the gauge symmetry.

The role of the zero-observable $O_P$ will be played by the trace of the Wilson loop in some representation:

\[ O_R^0 = \text{Tr}_R g(x), \quad g(x) = P \exp \oint_{L_x} A_t dt \]

(5.53)

Note that we have broken the $U(1)$ $R$-symmetry which counts the ghost number, since $\partial_t$ in the formula above is not rotating. In fact, it is not very surprising given an interpretation of the twisted theories.

The result is that if one restricts onto the time-independent $g = e^{\phi}$ then the observables are just the Donaldson descendents of $e^{\phi}$.
5.7.3. Elliptic genera

Let $x_i$ denote the eigenvalues of the skew-diagonalized curvature two form $\frac{R}{2\pi}$. We also denote

$$S_x V = 1 + xV + x^2 S^2 V + \cdots$$

$$\Lambda_x V = 1 + xV + x^2 \Lambda^2 V + \cdots$$  \hspace{1cm} (5.54)

Then the following identities hold [85]:

$$F(q) = q^{-d/16} \hat{A}(M) Ch(\otimes_{k \in \mathbb{N}} \Lambda_{q^k} T \otimes_{k \in \mathbb{N}} S_{q^k} T) =$$

$$\prod_{j=1}^{d/2} q^{-1/8} \frac{x_j/2}{\sinh(x_j/2)} \prod_{k \in \mathbb{N}} (1 + q^k e^{x_j}) (1 + q^k e^{-x_j})$$

$$q^{-d/4} \hat{A}(M) Ch(\otimes_{k \in \mathbb{N}} S_{q^k} T) =$$

$$\prod_{j=1}^{d/2} q^{-1/12} \frac{x_j/2}{\sinh(x_j/2)} \prod_{k \in \mathbb{N}} (1 - q^k e^{x_j}) (1 - q^k e^{-x_j})$$

$$q^{d/48} \prod_{j=1}^{d/2} \frac{x_j/2}{i \vartheta_1(x_j/2 \tau)}$$  \hspace{1cm} (5.56)

5.7.4. Characteristic classes and Theta-Functions

Theta function. Let $q = e^{2\pi i \tau}$, $\Im \tau > 0$,

$$\vartheta_1(x; \tau) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} q^{n^2} e^{2\pi i n(x + \frac{1}{2})}$$

The modular transformed quantities are: $\tilde{x} = -x/\tau$, $\tilde{\tau} = -1/\tau$, and

$$\vartheta_1(\tilde{x}; \tilde{\tau}) = i \sqrt{\tau} e^{i \pi x^2 \tau} \vartheta_1(x; \tau)$$

There is also a multiplicative representation, which is useful for our purposes:

$$\vartheta_1(x; \tau) = iq^{1/8} (e^{i \pi x} - e^{-i \pi x}) \prod_{n=1}^{\infty} (1 - q^n)(1 - q^n e^{2\pi i x})(1 - q^n e^{-2\pi i x})$$

Thus,

$$\eta(q)^3 = \frac{1}{2\pi} \vartheta_1'(0; \tau)$$

96
where Dedekind eta is given by the infinite product:

\[ \eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \]

Theta-function obeys the following transformation properties:

\[ \theta_1(x + \tau; \tau) = -q^{-\frac{1}{2}} e^{-2\pi i x} \theta_1(x; \tau), \]

\[ \theta_1(-x + \frac{1 + \tau}{2}; \tau) = e^{2\pi i x} \theta_1(x + \frac{1 + \tau}{2}) \]

Todd class, Chern character, Bernoulli numbers.

For \( E = \oplus_i L_i \), \( x_i = c_1(L_i) \), one has:

\[ Td(E) = \prod_i \frac{x_i}{1 - e^{-x_i}} \]

\[ Ch(E) = \sum_i e^{x_i} = \sum_n ch_n(E) \]

\[ c(E) = \prod_i (1 + x_i) = \sum_n c_n(E) \]  \hspace{1cm} (5.57)

In particular, Todd class of the four-dimensional manifold \( \Sigma \) is equal to:

\[ Td(\Sigma) = 1 + \frac{c_1}{2} + \frac{(c_1^2 + c_2)}{12} \]  \hspace{1cm} (5.58)

More generally,

\[ log(Td(E)) = \frac{1}{2} c_1(E) + \sum_{N=1}^{\infty} (-1)^N \zeta(2N)(2N)! \frac{c_{2N}}{(2\pi)^{2N}} \]  \hspace{1cm} (5.59)
6. \( N = 2 \) strings and holomorphic gravity

6.1. Target space theories of \( N = 2 \) strings

6.1.1. Closed string sector

\( N = 2 \) string target space theory was studied in [14], in the signature \((2, 2)\). It was shown, in particular, that the closed string sector is described by the four dimensional theory of a massless scalar \( \phi \), whose tree level action has the form:

\[
\int \omega_0 \wedge \partial \bar{\partial} \phi + \phi \partial \bar{\partial} \phi \wedge \partial \bar{\partial} \phi
\]

(6.1)

The way the scalar \( \phi \) enters the vertex operators implies that it corresponds to the deformations of a Kähler structure, \( \omega_0 \) being the reference Kähler form and \( \phi \) the deformation of the Kähler potential. The crucial property of this action is that all \( n \)-point functions vanish on-shell for \( n > 3 \) at the tree level in the \((2, 2)\) signature of the space-time. The equations of motion following from this action are

\[
\omega \wedge \omega = \omega_0 \wedge \omega_0
\]

for \( \omega = \omega_0 + \partial \bar{\partial} \phi \). These are Plebanski equations and they describe the Einstein Kähler metrics. Indeed, Ricci-flatness on a Kähler manifold is equivalent to the statement, that locally

\[
\det g_{ij} = f(z) \bar{f}(\bar{z})
\]

and therefore by a change of coordinates the determinant of metric can be set equal to one.

The action (6.1) is a particular case of the generalized WZW model, corresponding to the abelian group. Notice that one has to think of the Kähler potential as being the metric on the auxilliary line bundle \( L \) over \( \Sigma \). In fact, it is the prequantization bundle, as its first Chern class is proportional to the Kähler form \( \omega \). This proposes that \( N = 2 \) string has to do with the quantization of \( \Sigma \) in the sense of geometric quantization. The reason we emphasize that the Kähler metric should be thought of being the auxilliary data is that the intrinsically gravitational actions, which we write shortly, have \( \phi \) being the conformal factor rather then Kähler potential.
6.2. Gravitational sector of a holomorphic theory

In this section we will be brief. The investigations are still in progress so we will just sketch a few ideas. We want to study a higher dimensional generalization of two dimensional gravity with the emphasis on the holomorphic symmetries. The first thing to note is the dependence of the partition functions of the holomorphic theories we have considered on the metric on $\Sigma$. The standard by now procedure involves the consideration of the family of the metrics on $\Sigma$ or family of complex structures on $\Sigma$. It is convinient to combine all the members of the family in one complex manifold $X$, which is usually called a universal surface (for $\dim\Sigma = 2$). $X$ has a natural holomorphic map $\pi$ to some complex variety $S$, which (locally) parameterizes the complex structures on $\Sigma$ (up to isomorphisms). The fiber $\pi^{-1}(s) = \Sigma_s$ is a surface with the complex structure corresponding to $s$. It is the gravitational analogue of the universal bundle we had in the section 5??.

Now, the partition function of a chiral holomorphic theory whose fields are taking values in a holomorphic bundle $E$ over $\Sigma$ is a section of some line bundle $L$ over the moduli space $S$. The topology of this bundle is given again by the Riemann-Roch-Grothendieck formula:

$$Ch(L) = \int_{\Sigma} Td_\Sigma Ch(E)$$

(6.2)

The difference with the previous cases is that the classes entering $Td$ now have components along $S$.

Now we list a few examples, which will include two dimensional $b\alpha$-systems of spin $j$, four-dimensional $b\alpha$-systems, coupled to the bundle $\psi^j(T^{1,0}_\Sigma)$, where $\psi^j(T^{1,0}_\Sigma)$ is a virtual bundle, whose $k$'th Chern class $c_k$ is the one of $T^{1,0}_\Sigma$ (holomorphic tangent bundle), multiplied by $j^k$.

$$\begin{align*}
\dim\Sigma & = 1 \\
\dim\Sigma & = 2
\end{align*}$$

$$\begin{align*}
\frac{6j^2+6j+1}{12} c_1^2 & \\
\frac{j(j+1)(2j+1)}{12} c_1^3 + \frac{j(j+1)(6j^2-1)}{12} c_1 c_2
\end{align*}$$

(6.3)

In two dimensions the case $j = 1$ corresponds to the ghosts in the Polyakov's approach to the string theory. We see the famous number $13 = 26/2$ emerging from the (6.3). In four dimensions the case $j = 1$ also corresponds to the ghosts of the diffeomorphism group and they have two kinds of anomalies:

$$\frac{1}{2} c_1^3 + \frac{5}{6} c_1 c_2$$

99
Notice that the $j = -1$ case corresponds to the vanishing of both anomalous terms. It can be explained as follows. The $j = -1$ is the $bc$-system coupled to the bundle of $(1,0)$ forms. In particular, $e$ is a $(1,1)$ form, $b$ is $(1,0)$ - form. One can assign to the fields $e$ and $\phi$, $\bar{\phi}$ such types, that the measure on the fields will be well-defined without any use of the Kähler form $\omega$. Namely, the proper choice is $e$ being $(1,2)$, $\phi$ - $(1,0)$ and $\bar{\phi}$ - $(1,2)$. Then the metrics (or the mass term) on the space of fields is:

$$\int_{\Sigma} \overline{e} e + b e + \bar{\phi} \phi$$

Now let us consider what are the effective theories these anomalies lead to. In two dimensions it is known that the anomaly $c_2^1$ gives rise to the Liouville theory, whose field is the conformal factor of the metric and the action written in the background metric $\hat{g} = \hat{g}_{zz} dz d\bar{z}$ has the form:

$$S_{\hat{g}}(\phi) = \int_{\Sigma} \partial \phi \hat{\partial} \phi + 2 \hat{R} \phi$$

where $\hat{R} = \partial \bar{\partial} \log |\hat{g}_{zz}|^2$ is the curvature of $\hat{g}$. The action has the cocycle property:

$$S_{\hat{g}}(\phi + \phi') = S_{\hat{g}_{ee}}(\phi') + S_{\hat{g}}(\phi) \quad (6.4)$$

In four dimensions we have two anomalies $c_1^3$ and $c_1 c_2$. The first one gives rise to the action which depends again only on the conformal factor:

$$S_{g}(\phi) = \int_{\Sigma} \partial \phi \wedge \bar{\partial} \phi \wedge \partial \bar{\partial} \phi + 3 \hat{R} \wedge \hat{R} \phi + 3 \hat{R} \wedge \partial \bar{\partial} \phi \phi \quad (6.5)$$

and also has the cocycle property (6.4).

The second one depends on the metrics itself. One can write the action using the methods of the section 2.3.

Notice the similarity of the action (6.5) and the Plebanski gravity action.

Paradox We seem to find some kind of a gravitational anomaly in the four dimensional theory. It contradicts some well-known statements that the gravitational anomaly may only exist in $4k + 2$ dimensions [89]. The resolution is simple. The absence of anomalies follows from the fact that for $SO(2k)$ group the only characteristic classes have dimensions $4p$ (Pontryagin classes). On the other hand, we explicitly broken the Lorentz group down to $U(k)$ and therefore we have got the Chern classes which are present in any even dimension.
6.3. Equations of motion from the Chern-Simons theory

Here we present an alternative way of deriving the Plebanski and similar actions from the gauge theory. Consider a five-manifold $X_5 = X_4 \times I$, where $I$ as usual denotes the time interval $[0, 1]$. Let us assume that $X_4$ is a Kähler manifold and $\omega$ is its Kähler form. Consider the Chern-Simons theory with the action and a supersymmetry $Q$:

$$S = \int_{X_5} \omega \wedge dA + AF^2 + dt \wedge (\omega + F)\psi\psi$$

$$F = dA, QA = \psi, Q\psi = t_{\frac{dt}{\partial t}} F, t_{\frac{dt}{\partial t}} \psi = 0 \quad (6.6)$$

with $\psi$ being a fermionic one-form with only four components (as obvious from the last condition. Since $X_4$ is complex it makes sense to impose the second class constraints:

$$F^{2,0} = F^{0,2} = 0$$

(the proper way to do that was discussed in the chapter 5). If $X_5$ has no boundary then the action $S$ is annihilated by $Q$. In the presence of the boundary $X_4 \times \partial I$ $S$ can produce a boundary term when $Q$ is applied. Indeed:

$$QS = \int_{X_4 \times \{0\}} - \int_{X_4 \times \{1\}} \omega \wedge A\psi + AF\psi$$

We can impose the conditions $\psi|_{0,1} = 0$ which will ensure the invariance of the action.

Now, the standard localization argument shows that the theory is localized on the zero locus of the $Q$ transformations, i.e. on the solutions to the equations $\psi = 0, F_{\mu\nu} = 0$. Using the constraints the theory can be rewritten as follows: locally one can find such functions $\varphi$ and $\bar{\varphi}$, that $A_i(t) = \partial_t \varphi + A_i(0), A_{\bar{i}}(t) = \bar{\partial}_t \bar{\varphi} + A_{\bar{i}}(0)$, where $i$ is a holomorphic index while $\bar{i}$ is an anti-holomorphic one. Now, imposing the localization equation we would get:

$$\partial(\partial_t \varphi - A_t) = 0, \bar{\partial}(\partial_t \bar{\varphi} - A_t) = 0$$

and hence

$$\varphi = \varphi_0 + \int_0^t A_t dt, F = \bar{\partial}\partial\phi, \phi = \Re(\varphi), \partial_i \phi = 0$$

The action becomes:

$$\int_{X_4} \frac{1}{3} (\partial \bar{\partial} \phi)^2 + \phi \frac{1}{2} \partial \bar{\partial} \phi F + \phi \frac{1}{2} F^2$$

which is what we were aiming for.
6.4. Torsion free sheaves

This section is really speculative. Recall, that the space of non-abelian instantons is compactified by adding the point-like instantons, which corresponds on the level of bundles to the sheaves of a certain kind (they have no torsion as modules over the sheaf $\mathcal{O}$ of holomorphic functions). In some situations the same phenomenon occurs in abelian story, like it happened for Nakajima’s algebras [54], [20]. Previously we derived the target space theory equations of motion for $N=2$ closed string starting with the characteristic class $c_1^2$. It seems that for the $U(1)$ theory there is no other choice. But, if one assumes that the relevant objects are the torsion free sheaves of rank one, than they can have $c_2$, supported at finite set of points. This leads to two consequences. First, the $c_1c_2$ term gives by descend the insertions of the operators $e^b$ at the points, where $c_2$ is supported. Secondly, the non-abelian part of the descend action contains the terms which govern the motion of the points, where $c_2$ is supported. One has, therefore, a picture of a gas of 0-branes on $\Sigma$, interacting with the usual field $\phi$ in the string spectrum. It would be nice to develop this picture further and to compare it with [90].

6.5. One-loop finiteness of four dimensional avatar of $WZW_2$

In this thesis we have studied higher dimensional $WZW$ theories mostly as effective field theories arising from free models. It is relevant to mention the amusing on-shell one-loop finiteness of the theory (2.4).

The perturbative vacuum is one-loop finite. The one-loop renormalization of the vacuum state may be carried out via the background-field method. We let $g = e^{\pi/\sqrt{\pi}} g_{d}$ where $g_{d}$ is a solution of the classical equations of motion with given boundary conditions. For one-loop renormalization it suffices to keep the terms quadratic in $\pi(x)$:

$$S = S_{d} + \int (\nabla_{\mu}(J_{d})\pi)^2 + \pi^{a} M^{ab} \pi^{b} + \mathcal{O}(\pi^{3})$$

where the connection $\nabla_{\mu}(J_{d})$ and $M^{ab}$ are constructed from the classical solution of the equation of motion. The divergent terms at one-loop may be extracted from the Seeley expansion:

$$\Delta S = \int \frac{1}{c_4} C + \frac{1}{c_2} \text{Tr} M + (\log \epsilon) \text{Tr} \left[ \frac{1}{2} M^2 + \frac{1}{6} (\nabla^2 M - \frac{1}{2} F_{\mu\nu}(J)^2) \right] + \cdots$$

102
where $C$ is a $\pi$-independent constant and $F_{\mu\nu}(J)$ is a fieldstrength constructed from $\nabla_\mu(J_d)$. For the $WZW_4$ action we find that

$$M = 0, \quad \hat{\nabla}(J_d) = \bar{\partial}, \quad \nabla(J_d) = \partial + [g_d^{-1} \partial g_d, \cdot].$$

We conclude that there is no quadratic divergence at the Kähler point and in addition, using the equations of motion we see that the logarithmic divergences also vanish once boundary terms are properly included. Of course, these statements can also be checked directly using Feynman diagrams.

**Discussion of Renormalizability** Given the surprising one-loop finiteness one may naturally wonder if the model is finite. This remains a mystery. Several arguments indicate that the renormalizability properties of the theory are special, but eventually they remain inconclusive.

Even if the theory is 2-loop infinite we should not give up, given all the evidence above for the existence of a free field representation for the algebraic correlation functions.
7. Integrability and holomorphy

In this chapter we shall deal with the aspects of holomorphic theories which make contact with the integrable models of various kinds. We shall be mostly interested in the holomorphic Hamiltonian systems, whose phase spaces are the cotangent bundles of the moduli spaces of bundles. These systems have two faces.

One of them is related to the equations on the holomorphic blocks, like Knizhnik-Zamolodchikov [91] and Bernard [92] equations. In two dimensions these equations govern the behavior of the conformal blocks as the complex structure of the surface (perhaps, with puncture) is changing. At the critical level \( k = -h^V \) the derivative along the moduli space of complex structures disappears and the equations become the commuting differential operators on the moduli space of holomorphic bundles. There are no conformal blocks at the critical level, but the differential operators make sense (like \( \overline{\partial} \) operator might have no global kernel, but be well-defined). These operators can be interpreted as a quantization of a family of commuting Hamiltonians on the cotangent bundle to the moduli space [93].

Another face of the holomorphic integrable systems appeared recently in the study of low-energy effective actions for \( N = 2 \) supersymmetric gauge theories in four dimensions [94]. It turns out that the same commuting Hamiltonians can be identified with the order parameters in the gauge theory while the geometry of the space of vacua is encoded in the family of (Complex) Liouville tori for the model (when the integrable system is holomorphic, i.e. the symplectic form has the type \( (2, 0) \), the Hamiltonians are the holomorphic functions, then the Liouville integrability theory gets "complexified" and the torus of the real dimension equal to the half the dimension of a phase space gets replaced by an abelian variety, which is generically a Jacobian of an auxiliary curve, called a spectral curve).

We shall investigate the related models from the gauge theory perspective. The material here is not essentially new [95], but we included it for the sake of completeness. Also, at the time the results below were obtained their relevance to the supersymmetric gauge theory wasn’t known.

7.1. Two dimensional models: Hitchin systems and their degenerations

7.1.1. Holomorphic Bundles and Many-Body Systems

In this section we show that spin generalization of elliptic Calogero-Moser system, elliptic extension of Gaudin model and their cousins are the degenerations of Hitchin systems. We recall the construction of these systems, and then generalize them.
Integrable many-body systems attract attention for the following reasons: they are important in condensed matter physics and they appear quite often in two dimensional gauge theories as well as in conformal field theory. Recently they have been recognized in four dimensional gauge theories.

Among these systems the following ones will be of special interest for us:

1. **Spin generalization of Elliptic Calogero - Moser model** - it describes the system of particles in one (complex) dimension, interacting through the pair-wise potential. The explicit form of the Hamiltonian is:

   \[ H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{i \neq j} \text{Tr}(S_i S_j) \varphi(z_i - z_j) \]

   where \( z_i \) are the positions of the particles, \( p_i \) - corresponding momenta and \( S_i \) are the "spins" - \( l \times l \) matrices, acting in some auxiliary space. The conditions on \( S_i \) will be specified later. The only point to be mentioned is that the Poisson brackets between \( p, z, S \) are the following:

   \[ \{ p_i, z_j \} = \delta_{ij} \]

   \[ \{(S_i)_{ab}, (S_j)_{cd}\} = \delta_{ij} (\delta_{ad} (S_i)_{bc} - \delta_{bc} (S_i)_{ad}) \]

2. **Gaudin model and its elliptic counterpart.** We describe first the rational case. Consider a collection of \( L \) points on \( \mathbb{P}^1 \) in generic position: \( w_1, \ldots, w_L \), assign to each \( w_a \) a spin \( T_a \) (an \( N \times N \) matrix) and define the Hamiltonians [96]:

   \[ H_a = \sum_{b \neq a} \frac{\text{Tr}(T_a T_b)}{(w_a - w_b)} \]

   The main goal of this note is to include these two (seemingly) unrelated models in the universal family of integrable models, naturally related to the moduli spaces of holomorphic bundles over the curves. It turns out, that the appropriate objects to study are Hitchin systems. As a by-product we invent elliptic Gaudin model, which includes both cases as a special limits. We shall also obtain a prescription for construction of integrals of motion and action-angle variables.

   We must confess that all the models we are discussing are motivated by the studies of Knizhnik-Zamolodchikov-Bernard equations [91], [92], [97], [98], [99]. One of the outcomes of our work might be an insight in the \( \mathcal{W} \)- generalizations of them.
Hitchin systems. Hitchin has introduced in [100] a family of integrable systems. The phase space of these systems can be identified with the cotangent bundle $T^*\mathcal{N}$ to the moduli space $\mathcal{N}$ of stable holomorphic vector bundles of rank $N$ (for $GL_N(\mathbb{C})$ case) over the compact smooth Riemann surface $\Sigma$ of genus $g > 1$. His construction can be briefly described as follows. Fix the topological class of the bundles (i.e. let us consider the bundles $\mathcal{E}$ with $c_1(\mathcal{E}) = k$, with $k$ - fixed). Consider the space $\mathcal{A}^s$ of stable complex structures in a given smooth vector bundle $V$, whose fiber is isomorphic to $\mathcal{O}^N$. The notion of stable bundle comes from geometric invariants theory and implies in this context, that for any proper subbundle $U$:

$$\frac{\deg(U)}{rk(U)} < \frac{\deg(V)}{rk(V)}$$

The quotient of $\mathcal{A}^s/\mathcal{G}$ of the space of all stable complex structures by the gauge group is the moduli space $\mathcal{N}$. Its dimension is given by the Riemann-Roch theorem

$$\dim(\mathcal{N}) = N^2(g - 1) + 1$$

Now consider a cotangent bundle to $\mathcal{A}^s$. It is the space of pairs:

$$\phi, d''_A$$

where $\phi$ is a $Mat_N(\mathbb{C})$ valued $(1,0)$ - differential on $\Sigma$, $d''_A$ is an operator, defining the complex structure on $V$:

$$d''_A : \Omega^0(\Sigma, V) \to \Omega^{0,1}(\Sigma, V)$$

The field $\phi$ is called a Higgs field and the pair $d''_A, \phi$ defines what is called a Higgs bundle. In the framework of conformal field theory the Higgs field is usually referred to as the holomorphic current, while holomorphic bundle defines a background gauge field.

The cotangent bundle $T^*\mathcal{A}^s$ can be endowed with a holomorphic symplectic form:

$$\omega = \int_\Sigma \text{Tr} \delta \phi \wedge \delta d''_A$$

where $\delta d''_A$ can be identified with a $(0,1)$ - form with values in $N$ by $N$ matrices. Gauge group $\mathcal{G}$ act on $T^*\mathcal{A}^s$ by the transformations:

$$\phi \to g^{-1} \phi g$$

$$d''_A \to g^{-1} d''_A g$$

106
and preserves the form $\omega$. Therefore, a moment map is defined:

$$\mu = [d'_{A^s}, \phi]$$

Taking the zero level of the moment map and factorizing it along the orbits of $G$ we get the symplectic quotient, which can be identified with $T^{*}N$. Now the Hitchin Hamiltonians are constructed with the help of holomorphic $(1-j,1)$-differentials $\nu_{j,i,j}$ where $i_j$ labels a basis in the linear space

$$H^1(\Sigma, K \otimes T^j) = \mathbb{C}^{(2j-1)(g-1)}$$

for $j > 1$ and $\mathbb{C}^g$ for $j = 1$. Take a gauge invariant $(j,0)$-differential $\text{Tr} \phi^j$ and integrate it over $\Sigma$ with the weight $\nu_{j,i,j}$:

$$H_{j,i,j} = \int_{\Sigma} \nu_{j,i,j} \text{Tr} \phi^j$$

Obviously, on $T^{*}A^s$ these functions Poisson-commute. Since they are gauge invariant, they will Poisson-commute after reduction. Also it is obvious, that they are functionally independent and their total number is equal to

$$g + \sum_{j=2}^{N} (2j-1)(g-1) = N^2(g-1) + 1 = \text{dim}(N)$$

Therefore, we have an integrable system.

**Holomorphic bundles over degenerate curves** Now let us consider a degeneration of the curve. Recall, that the normalization of the stable curve $\Sigma$ is a collection of a smooth curves $\Sigma_\alpha$ with possible marked points, such that any component of genus zero has at least three marked points and every component of genus one has at least one such a point. For each component $\Sigma_\alpha$ we have a subset $X_\alpha = \{x_1^\alpha, \ldots, x_L^\alpha\}$ of points. Let us denote the pair $(\Sigma_\alpha, X_\alpha)$ as $C_\alpha$. The disjoint union of $C_\alpha$’s is mapped onto $\Sigma$ by the normalization map $\pi$. Let us denote by $X_{\alpha\beta}$ the set of double points $\pi(X_\alpha) \cap \pi(X_\beta)$ for $\alpha \neq \beta$ and as $X_{\alpha\alpha}$ the set of double points in $\pi(X_\alpha)$ (these appear due to pinching the handles). The union of all $X_{\alpha\beta}$ we shall denote by $X \subset \Sigma$. We define $x^{ij}_{\alpha\beta} \in X_{\alpha\beta}$ as $\pi(x^i_\alpha) \cap \pi(x^j_\beta)$. Notice, that it may be empty. A stable bundle $E$ over $\Sigma$ is a collection of holomorphic bundles $E_\alpha$ over $\Sigma_\alpha$ of rank $N$ (there might be some generalizations with different ranks of the bundle over different components - these are unnatural as a degeneration of the bundle over smooth curve) with the identifications $g_{i\alpha\beta}^{ij}$ of the fibers

$$g_{i\alpha\beta}^{ij} : E_\alpha |_{x^{ij}_{\alpha\beta}} \rightarrow E_\beta |_{x^{ij}_{\beta}}$$

107
with the obvious condition: \( g_{\alpha\beta}^{ij} g_{\beta\alpha}^{ji} = 1 \).

The gauge group acts on the complex structure of the bundle \( \mathcal{E}_\alpha \) for each \( \alpha \) as in the smooth curve case. The new ingredient is the action on \( g_{\alpha\beta}^{ij} \). Fix a gauge transformation \( g_\alpha \) for each component of \( \Sigma \). Then \( g_{\alpha\beta}^{ij} \) are acted on by \( g_\alpha \) as follows:

\[
g_{\alpha\beta}^{ij} \rightarrow g_\beta(x_\beta^i)^{-1} g_{\alpha\beta}^{ij} g_\alpha(x_\alpha^i)
\]

Now we have to introduce a notion of stable bundle. The condition of stability is:

For each collection of proper subbundles \( \mathcal{F}_\alpha \subset \mathcal{E}_\alpha \), such that

\[
g_{\alpha\beta}^{ij}(\mathcal{F}_\alpha|_{x_\alpha^i}) = \mathcal{F}_\beta|_{x_\beta^i}
\]

and

\[rk(\mathcal{F}_\alpha) = N' < N \]

for each \( \alpha \) the following inequality holds:

\[
\text{deg}(\mathcal{F}_\alpha) < \frac{N'}{N} \text{deg}(\mathcal{E}_\alpha)
\]

for any \( \alpha \).

Let \( \mathcal{A} \) denote the space of collections of \( d'_{A,\alpha} \) operators in each \( \mathcal{E}_\alpha \) together with \( g_{\alpha\beta}^{ij} \) for each \( \alpha \) and \( \beta \). Let \( \mathcal{A}^s \) denote the subspace of \( \mathcal{A} \), consisting of stable objects. The cotangent bundle \( T^*\mathcal{A}^s \) can be identified with the space of collections of pairs

\[(\mathcal{E}_\alpha, \phi_\alpha), \phi_\alpha \in \Omega^{1,0}(\Sigma_\alpha) \otimes \text{End}(\mathcal{E}_\alpha)\]

and

\[(g_{\alpha\beta}^{ij}, p_{\alpha\beta}^{ij}), p_{\alpha\beta}^{ij} \in T_{g_{\alpha\beta}^{ij}}^* \text{Hom}(\mathcal{E}_\alpha|_{x_\alpha^i}, \mathcal{E}_\beta|_{x_\beta^j})\]

We normalize \( p_{\alpha\beta}^{ij} \): \( p_{\alpha\beta}^{ij} = -Ad^*(g_{\alpha\beta}^{ij})p_{\beta\alpha}^{ji} \). The Higgs fields \( \phi_\alpha \) are allowed to have singularities at the marked points. As we will see, they could have poles there. Now we shall proceed as in the previous section. Consider the gauge group action on \( T^*\mathcal{A}^s \). Since the gauge group \( \mathcal{G} \) is essentially the product of gauge groups \( \mathcal{G}_\alpha \), the moment map is a collection of the moment maps for each component \( \Sigma_\alpha \):

\[\mu_\alpha = [d'_{A,\alpha}, \phi_\alpha] + \sum_{\beta, i, j} p_{\alpha\beta}^{ij} \delta^j_\alpha(x_\alpha^i)\]

108
where the sum over $i$ runs from 1 up to $L_\alpha$ while $\beta$ and $j$ are determined from the condition, that $\pi(x_{ij}^\alpha) = \pi(x_{ij}^\beta)$. Let us now repeat the procedure of reduction. At the first step we should restrict ourselves onto the zero level of the moment map. It means, that $\phi_\alpha$ becomes a meromorphic section of the bundle $\text{End}(\mathcal{E}_\alpha) \otimes \Omega^{1,0}(\Sigma_\alpha)$ with the first order poles at the double points. The residue of $\phi_\alpha$ at the point $x_{ij}^\alpha$ equals to $\nu_\alpha^{ij\beta}$ for appropriate $\beta, j$. This condition is compatible with the definition of the canonical bundle over the stable curve.

On the next step we take a quotient with respect to the gauge group action and get the reduced space $T^*\mathcal{N}$. The space $\mathcal{N}$ is the quotient of $\mathcal{A}^g$ by $\mathcal{G}$. The symplectic form on $T^*\mathcal{N}$ can be written as:

$$\omega = \sum_\alpha \omega_\alpha + \sum_{(\alpha, \beta)(\beta, j)} \text{Tr}(g^{ij}_{\beta\alpha} p^{ij}_{\alpha\beta}) \wedge \delta g^{ij}_{\alpha\beta}$$

Let us calculate the dimension of $T^*\mathcal{N}$. We shall calculate the (complex) dimension of $\mathcal{N}$ by means of the following trick. The moduli space $\mathcal{N}$ can be projected onto the direct product of moduli spaces $\mathcal{N}_\alpha$ of the stable bundles over $\Sigma_\alpha$’s. Actually, the map is to the product of the moduli of holomorphic bundles, but the open dense subset, consisting of the stable bundles is covered. The projection simply takes the collection of $\mathcal{E}_\alpha$’s to the product of equivalence classes in $\mathcal{N}_\alpha$’s. The fiber of this map can be identified with the quotient $G/H$, where $G$ is the set of collections of $g^{ij}_{\alpha\beta}$, while $H$ is the group of automorphisms of $\times_\alpha \mathcal{E}_\alpha$. This group is a product over all components $\Sigma_\alpha$ of the groups $H_\alpha$. For the genus zero component $H_\alpha$ is $G\text{L}_N(\mathbb{C})$, while genus one component provides a maximal torus $- (\mathbb{C}^*)^N$. The higher genus components give $\mathbb{C}^g$ which acts as a trivial automorphism.

Therefore, at generic point, we conclude, that the dimension of $\mathcal{N}$ is

$$\text{dim}(\mathcal{N}) = \sum_\alpha \text{dim}(\mathcal{N}_\alpha) + \text{dim}(G/H) =$$

$$= N^2 E(\Sigma) + \sum_\alpha N^2 (g(\Sigma_\alpha) - 1) = N^2 (g - 1) + 1$$

(7.1)

where we have used Riemann-Roch theorem in the form

$$\text{dim}(\mathcal{N}_\alpha) - \text{dim}(H_\alpha) = N^2 (g(\Sigma_\alpha) - 1),$$

$E(\Sigma)$ is the total number of double points.

Hamiltonian systems on $T^*\mathcal{N}$. Now we shall define the Hamiltonians. For each $\alpha$ we take $\nu_{\alpha,l,k}$ - the $k$'th holomorphic $(1-l,1)$ differential on $\Sigma_\alpha - X_\alpha$ and construct a holomorphic function on $T^*\mathcal{A}^g$:

$$H_{\alpha,l,k} = \int_{\Sigma_\alpha} \nu_{\alpha,l,k} \text{Tr}(\phi_\alpha^l)$$

Obviously, all $H_{\alpha,l,k}$ descend to $T^*\mathcal{N}$ and Poisson-commute there.

109
7.1.2. Gaudin model, Spin Elliptic Calogero-Moser System and so on ...

Genus zero models. Consider a component of genus zero. Let us describe explicitly the part of $T^* N$ related to this component as well as the Hamiltonians. We shall omit the label $\alpha$ in the subsequent formulas to save the print. Thus, we have $C = (\mathbb{P}^1, x_1, \ldots, x_L)$.

Assume first, that for $a \neq b$, $\pi(x_a) \neq \pi(x_b)$. Then $p_{\alpha \beta}^b$ can be denoted simply as $T_a$ without any confusion. There are no continuous moduli of stable holomorphic bundles over the sphere. For simplicity we consider the case when $\mathcal{E}$ is trivial and therefore, we can assume that $d'_{\alpha}^b$ is just the $\bar{\partial}$ operator. The moment map condition:

$$0 = \bar{\partial}\phi + \sum_a T_a \delta^2(x_a)$$

is easily solved:

$$\phi(x) = \frac{1}{2\pi\sqrt{-1}} \sum_a \frac{T_a}{(x - x_a)}$$

Notice, however, that every trivial holomorphic bundle over $\mathbb{P}^1$ has an automorphism group $GL_N(\mathbb{C})$, which acts nontrivially on $\phi$ as well as on $T_i$. In fact, the reduction with respect to this subgroup is forced by our equation: the sum of residues of $\phi$ must vanish, giving rise to the constraint: $\sum_a T_a = 0$ which is nothing but the moment map for the $GL_N(\mathbb{C})$ action.

Our Hamiltonians in this case boil down to

$$H_{b,a,b} = \text{Res}_{x_a}(x - x_a)^{b-1} \text{Tr}^b(x),$$

where $1 \leq b \leq l, 1 \leq a \leq L$. These Hamiltonians (essentially $H_{2,a,2}$) are called Gaudin Hamiltonians$^{28}$

$$H_{1,a,1} = \text{Tr}(T_a), H_{2,a,1} = \text{Tr}(T_a^2),$$

$$H_{2,a,2} = \sum_{b \neq a} \frac{\text{Tr}(T_b T_a)}{(x_b - x_a)}$$

etc.

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$^{28}$ Historically, these Hamiltonians in the classical context were introduced by R. Garnier [101], and L. Schlesinger [102]. Nowadays, by Gaudin system one understands the quantum counterpart of this model, when $T_a$’s are replaced by the generators of the Lie algebra, acting in representation, attached to the point $x_a$.
Spin Calogero-Moser and Rational Ruijsenaars Systems. Now consider genus zero component $\Sigma_\alpha$ with double points. Let us decompose the set of marked points $X_\alpha$ as

$$X_\alpha = S \cup T \cup \sigma(T)$$  \hspace{1cm} (7.2)

where $t \in T$ and $\sigma(t) \in \sigma(T)$ are mapped to $x_{\alpha t}^{t \sigma(t)}$, while the restriction of $\pi$ on $S$ is surjective. We denote $p_{\alpha t}^{t \sigma(t)} = p_t, g_{\alpha t}^{t \sigma(t)} = g_t$. We have $p_{\alpha t}^{t \sigma(t)} = -Ad^*(g_t)p_t$. Solving the moment map condition as before, we get:

$$\phi_\alpha(x) = \sum_s \frac{p_s}{x-x_s} + \sum_t \frac{p_t}{x-x_t} - \frac{Ad^*(g_t)p_t}{x-x_{t \sigma(t)}}$$

Sum of the residues vanishes, giving rise to the moment equation:

$$\sum_s p_s + \sum_t (p_t - Ad^*(g_t)p_t) = 0$$  \hspace{1cm} (7.3)

This is the moment with respect to the group $GL_N(\mathbb{C})$ of automorphisms which acts on the data $(p_s, p_t, g_t)$ as follows:

$$g_t \rightarrow g^{-1}g_t, p_t \rightarrow g^{-1}p_t g, p_s \rightarrow g^{-1}p_s g$$

Now let us specialize to the case, when $\#T = 1$. The moment map condition will give

$$p_t - Ad^*(g_t)p_t = -\sum_s p_s$$

The phase space is the quotient of the manifold of $p_t, p_s, g_t$, satisfying this condition by the action of $GL_N(\mathbb{C})$. We have two options: either we parameterize the quotient by the conjugacy class of $g_t$, or by the one of $p_t$.

Consider the first option. Generically one can diagonalize $g_t$, and there will be left a group of diagonal matrices, which will act nontrivially on $p_t$ and $p_s$'s. Let us denote the eigenvalues of $g_t$ by

$$g_t \sim diag(e^{z_1}, \ldots, e^{z_N})$$

Then in the basis, where $g_t$ is diagonal, $p_t$ has a form:

$$(p_t)_{ij} = p_0 \delta_{ij} + \sum_s \frac{(p_s)_{ij}}{1 - e^{z_i - z_j}}$$
with the further condition
\[ \sum_s (p_s)_{ii} = 0 \] (7.4)
for any \( i \). This condition has an elliptic nature, as we will see, and has a very natural origin: the double point on the sphere comes from the pinching the handle.

Explicitely calculating \( H_{2,t,2} \) we get:
\[
H_{2,t,2} = \sum_i \frac{p_i^2}{2} + \sum_{i,j} \frac{\Sigma_{ij}}{\sinh^2(\frac{r_i-r_j}{2})}
\]
with
\[
\Sigma_{ij} = \sum_{s,s'} (p_s)_{ij} (p_{s'})_{ji}
\]

This Hamiltonian describes the particles with spin interaction. In the section 4.3. we shall represent the coupling \( \Sigma_{ij} \) as \( \text{Tr} S_i S_j \), which is the form of spin interaction we advertised in introduction. This model for \( \#S = 1 \) is the Spin generalization of Sutherland model §.

Now let us investigate another option - namely, we diagonalize \( p_t \). For simplicity we shall treat the case \( \#S = 1 \). We have:
\[
p_t = \text{diag}(\theta_1, \ldots, \theta_N)
\]
and
\[
(g_t)_{ij}(\theta_i - \theta_j) = (\tilde{p}_s)_{ij}
\]
where \( \tilde{p}_s = g_t p_s \). Now we make a further assumption. Suppose, that for some \( \nu \in \mathbb{C}^* \) the matrix \( p_s - \nu \text{Id} \) has rank one.

Then,
\[
p_s = \nu \text{Id} + u \otimes v
\]
where \( v \in (\mathbb{C}^N)^*, u \in \mathbb{C}^N. \)

Therefore, we can solve for \( g_t \):
\[
(g_t)_{ij} = \frac{\tilde{u}_i v_j}{\theta_i - \theta_j - \nu}
\]
and \( \tilde{u}_i = (g_t u)_i \). The consistency requires a linear equation:
\[
\sum_j \frac{u_j v_j}{\theta_i - \theta_j - \nu} = 1
\]
for any $i$. The solution is
\[ u_i v_i = \frac{P(\theta_i + \nu)}{P'(\theta_i)} \]
where $P(\theta) = \prod(\theta - \theta_i)$.

Finally we introduce the coordinates $z_i$, defined by
\[ e^{z_i} u_i = \tilde{u}_i \]

The Hamiltonians we can consider in this approach are the characters of $g_u$. We have:
\[ \text{Tr} g^k_I = \sum_{I \subseteq \{1, \ldots, N\}, \# I = k} e^{z_I} \prod_{i \in I, j \not\in I} \frac{\theta_{ij} + \nu}{\theta_{ij}}, \]
\[ z_I = \sum_{i \in I} z_i, \theta_{ij} = \theta_i - \theta_j, k = 1, \ldots, N \quad (7.5) \]

The system we get is the classical limit of Rational Macdonald system. If, instead of taking $G = GL_N(\mathbb{C})$ we would consider $SU(N)$, we will get what is called a rational Ruijsenaars-Schneider model [103],[104],[105].

It is clear, that without the simplifying assumption that $p_\alpha$ has the form of the scalar matrix plus the matrix of rank one, we would get the Spin Generalization of Ruijsenaars-Schneider Model. Conjecturally, our reduction provides a Hamiltonian description for the spin generalization of Ruijsenaars-Schneider model, considered in [106] (for its rational limit).

7.1.3. Elliptic models

The next interesting example is the genus one component. Again we omit label $\alpha$ and $p^{ij}_{\alpha \beta}$ gets replaced by $p_{ij}$. Generic holomorphic bundles over the torus are decomposable into the direct sum of the line bundles:
\[ \mathcal{E} = \bigoplus_{i=1}^{N} \mathcal{L}_i \]
Therefore, the moduli space $N_\alpha$ can be identified with the $N$'th power of the Jacobian of the curve, divided by the action of the permutation group. Let us introduce the coordinates $z_1, \ldots, z_N$ on $N_\alpha$. They are defined up to the elliptic affine Weyl group action. Let $\tau$ be the modular parameter of the elliptic curve. Then there are shifts
\[ z_i \to z_i + 2\pi \sqrt{-1} \frac{m_i \tau + n_i}{\tau - \tau} \]
with $m_i, n_i \in \mathbb{Z}$, induced by the gauge transformations

$$\exp(diag(2\pi \sqrt{-1}n_i(x - \bar{x}) + m_i(x\tau - \bar{x}\bar{\tau}))$$

as well as permutations of $z_i$'s. Up to these equivalencies $z_i$'s are the honest coordinates. **No double points.** First, we dispose of the case, when $\pi$ doesn’t map two points to one. Now the moment map condition is

$$\bar{\partial} \phi_{ij} + (z_i - z_j)\phi_{ij} + \sum_a (T_a)_{ij} \delta^2(x_a) = 0$$

The solution of this equation yields a Lax operator $\phi$:

$$i \neq j :$$

$$\phi_{ij} = \frac{\exp(z_{ij} \frac{x - \bar{x}}{\tau - \bar{\tau}})}{2\pi \sqrt{-1}} \sum_a (T_a)_{ij} \frac{\sigma(z_{ij} + x - x_a)}{\sigma(z_{ij}) \sigma(x - x_a)} e^{z_{ij} \frac{x - x_a}{\tau - \bar{x}}}$$

$$i = j :$$

$$\phi_{ii} = w_i + \frac{1}{2\pi \sqrt{-1}} \sum_a (T_a)_{ii} \zeta(x - x_a),$$

where we have denoted for brevity: $z_{ij} = z_i - z_j$. In these formulas $\sigma$ and $\zeta$ are Weierstrass elliptic functions for the curve with periods 1 and $\tau$. Since the sum of residues of meromorphic form $\phi_{ii}$ vanishes, we get the following equation:

$$\sum_a (p_a)_{ii} = 0$$

$$i = 1, \ldots, N,$$

which coincides with the moment for the maximal torus action. When the elliptic curve degenerates down to the rational one with the double point this equation becomes just (7.4).

Now we can compute our Hamiltonians. We have:

$$-4\pi^2 \text{Tr} \phi^2(x) =$$

$$\sum_i (w_i + \sum_a (T_a)_{ii} \zeta(x - x_a))^2 -$$

$$\sum_{i \neq j, a, b} (T_a)_{ij} (T_b)_{ji} \frac{\sigma(z_{ij} + x - x_a)\sigma(z_{ji} + x - x_b)}{\sigma(z_{ij})^2 \sigma(x - x_a) \sigma(x - x_b)} e^{z_{ij} \frac{x - x_a}{\tau - \bar{x}}},$$

where $x_{ab} = x_a - x_b$.  

114
Expanding this expression as:

\[-4\pi^2 \text{Tr} \phi^2(x) = \left( \sum_a \phi(x - x_a) H_{2,2,a} + \zeta(x - x_a) H_{2,1,a} \right) + H_{2,0}\]

we obtain:

\[H_{2,2,a} = \text{Tr} T_a^2\]  \hspace{1cm} (7.9)

as it could be guessed,

\[H_{2,1,a} = \sum_i w_i(T_a)_{ii} + \sum_{b \neq a;i} (T_a)_{ii}(T_b)_{ii} \zeta(x_{ab}) + \]

\[+ \sum_{b \neq a;i \neq j} \varepsilon^{z_{ij}(x_a - x_b)}(T_b)_{ij}(T_a)_{ji} \frac{\sigma(z_{ij} + x_{ab})}{\sigma(z_{ij})\sigma(x_{ab})} e^{z_{ij} \frac{x_{ab} - z_{ab}}{z_{ab} - x_{ab}}}\]  \hspace{1cm} (7.10)

These Hamiltonians will be called \textit{Elliptic Gaudin Hamiltonians}. The next interesting Hamiltonian is:

\[H_{2,0} = \sum_i w_i^2 + \sum_{i \neq j} (T_a)_{ij}(T_a)_{ji} \phi(z_{ij}) + \]

\[+ \sum_{i:a \neq b} (T_a)_{ii}(T_b)_{ii} \zeta(x_{ab}) - \zeta^2(x_{ab})) + \]

\[+ \sum_{i \neq j:a \neq b} \varepsilon^{z_{ij}x_{ab}}(T_b)_{ij}(T_a)_{ji} e^{z_{ij} \frac{x_{ab} - z_{ab}}{x_{ab} - z_{ab}}} \frac{\sigma(z_{ij} + x_{ab})}{\sigma(z_{ij})\sigma(x_{ab})} (\zeta(x_{ab} + z_{ij}) - \zeta(z_{ij})).\]  \hspace{1cm} (7.11)

Higher Hamiltonians provide us with the rest of the integrals of motion of this model.

\textbf{Double points.} In the case, when there are the double points, the formulas for the Lax operator and Hamiltonians are nearly the same, the only difference is in the condition on the \(T_a\)'s. It is easy to see, that the condition is: let us decompose the set of marked points as in (7.2) and introduce the notations \(p_t, p_s\) as before. Then the elliptic analogue of (7.3) is

\[\left[ \sum_s p_s + \sum_t (p_t - \text{Ad}^* r(g_t)p_t) \right]_{ii} = 0\]  \hspace{1cm} (7.12)

\[i = 1, \ldots, N\]

\textbf{Parabolic Structures, Spins and Coadjoint Orbits.} In this section we shall explain the relation of our moduli space to the moduli spaces of bundles with parabolic structures (this is what one usually expects to be considered on the punctured curve). Then we shall map the notations \(p_a\) for the Lie algebra elements to the spin notations \(S_t\), which were used in the beginning of the paper.
First of all, the integrable systems we have defined have an obvious invariant subvariety: the conjugacy classes of all \( T_a \)'s are conserved. Indeed, since \( \phi \) has a pole at \( x_a \) with the residue \( T_a \), then near \( x_a \)

\[
\text{Tr}\phi^n(x) \sim \frac{\rho_{a,n}}{(x-x_a)^n}
\]

(7.13)

where \( \rho_{a,n} \) equals:

\[
\rho_{a,n} = \text{Tr}T_a^n,
\]

hence the trace \( \text{Tr}T^n_a \) is a constant of motion. Therefore, each \( T_a \) will represent a point on the coadjoint orbit \( O_a \) of \( SL_N(\mathbb{C}) \). This orbit (which is generically diffeomorphic to the cotangent bundle to the compact flag variety) defines a parabolic structure at the point \( x_a \) (which is a fixed flag in the fiber over it).

In fact, the flag structure can be decoded from the \( T_a \) with the help of the following construction.

For simplicity, we consider the case without double points on the component \( C_a \). Again for simplicity we assume, that this coadjoint orbit is of the generic type.

Fix a point \( x_a \) and denote \( T_a \) simply as \( p \). Introduce a sequence of vector spaces

\[ \mathcal{E}^r, \ldots, \mathcal{E}^0 \]

Let \( d^i = \text{dim}\mathcal{E}^i \), we assume that \( d^i > d^{i+1} \). Consider the space of operators

\[ U^i : \mathcal{E}^i \to \mathcal{E}^{i+1}, V^i : \mathcal{E}^{i+1} \to \mathcal{E}^i \]

with the canonical symplectic form:

\[
\sum_i \text{Tr} \delta U^i \wedge \delta V^i
\]

This form is invariant under the action of the group

\[ H = \times_{i=1}^r \text{GL}(\mathcal{E}^i) \]

by the changes of bases. Therefore, one can make a Hamiltonian reduction at some central level of the moment map. Formally it amounts to imposing the constraints:

\[
U^{r-1}V^{r-1} = \zeta^r \text{Id}_{\mathcal{E}^r}
\]

\[
U^{i-1}V^{i-1} - V^iU^i = \zeta^i \text{Id}_{\mathcal{E}^i}
\]

116
for $i = 1, \ldots, r - 1$. Here the complex numbers $\zeta^i$ are related to the eigenvalues $\lambda_i$ of the matrix $\rho$ via:

$$\lambda_i - \lambda_{i-1} = \zeta^i, \lambda^{i-1} = 0$$

the multiplicity of the eigenvalue $\lambda^i$ equals $d^i - d^{i+1}$.

Finally, $\rho = V^0U^0 + \zeta^0\text{Id}_{\mathcal{E}^0}$.

The flag

$$\mathcal{F}^r \subset \ldots \subset \mathcal{F}^0 = \mathcal{E}^0$$

is constructed as:

$$\mathcal{F}^i = \text{Im}(V^0 \ldots V^{i-1}) : \mathcal{E}^i \to \mathcal{E}^0$$

Now for each point $x_a$ the orbit of $p_a$ can be represented with the help of the set of matrices $U^a, V^a$ and the numbers $\zeta^a$ such that $p_a = V^0U^0 + \zeta^0\text{Id}$. Here $a$ is a multiindex, including the numbers of the components of curves and the numbers of the points there.

Then for $i \neq j$ we could replace

$$(T_a)_{ij}(T_a)_{ji} \to Tr_{\mathcal{F}^a} S_i^a S_j^a$$

$$(S_i^a)_{\gamma \delta} = (U^a)_i^\gamma (U^a)^\delta_j$$

(7.14)

Hence, for the sphere with one double point and one extra puncture and for the elliptic curve with one puncture we get precisely the Spin Generalization of Calogero-Moser-Sutherland Model.

If number of punctures ($\#S$) is greater then one, then representation of the coupling as the product of spins of the type, just described, is not convenient. For the products $p_a p_b$ with $a \neq b$ there is no such interpretation. Therefore, $S_I$ operators in the Gaudin model as we described them in the begining of this section are just the matrices $p_a$.

The global consequence of this presentation is that on can view the moduli space of bundles over the degenerate curve as a family of products over the set of components of the curve of the moduli of bundles with parabolic structures. The restriction on the parabolic structures at the glued points is that the weights $\zeta^i$ and dimensions $d^i$ should coincide for both components, glued at the point.
7.1.4. Čech versus Dolbeau descriptions

Consider the Hitchin system for the compact curve $\Sigma$. We want to describe it in the language of Čech cocycles. Introduce local trivializations $g_\alpha$ like in chapter 4 and write $\phi_\alpha = g_\alpha^{-1} \phi g_\alpha$. We claim that the symplectic structure on the space $T^* M$ can be written as

$$\Omega = \delta \sum_{\alpha, \beta} \int_{\Gamma_{\alpha, \beta}} \text{Tr}(\phi_\alpha \partial g_{\alpha \beta} g_{\alpha \beta}^{-1})$$

The Čech description of Hitchin Hamiltonians is as follows. Introduce the ($-n$)-differentials $\epsilon_{\alpha \beta}$ which are holomorphic near $\Gamma_{\alpha, \beta}$. We will consider two ($-n$)-differentials $\epsilon_{\alpha \beta}$ and $\epsilon'_{\alpha \beta}$ to be equivalent if they differ by an exact cocycle, i.e.

$$\epsilon'_{\alpha \beta} = \epsilon_{\alpha \beta} + \epsilon_{\alpha} - \epsilon_{\beta}$$

where $\epsilon_{\alpha}$ is a holomorphic ($-n$) differential on $U_{\alpha}$. Then the Hitchin’s hamiltonians are given by the formulae:

$$H_\epsilon = \sum_{\alpha, \beta} \int_{\Gamma_{\alpha, \beta}} \text{Tr}(\epsilon_{\alpha \beta} \phi_{\alpha}^{n+1})$$

A simple contour deformation argument shows that the hamiltonian $H_\epsilon$ is not changed when $\epsilon$ is replaced by the equivalent $\epsilon'$.

It is clear now how to visualize the motion on $T^* M$ which is generated by the Hitchin hamiltonians.

Formulas for general case - genus $g$ curve with $L$ punctures. In this section we consider only one component $\Sigma$ of the stable curve. We assume that it has genus $g$ and $L$ punctures. We also assume, that $\Sigma$ has no double points.

Using the formulas for the twisted meromorphic forms on the curve, quoted in [99], we can easily write down the formula for the solution of the main equation

$$[d'_{A}, \phi] + \sum_i p_i \delta^2(x_i) = 0$$

In order to do this, we choose the following coordinatization of the moduli space $\mathcal{N}$ of holomorphic bundles over $\Sigma$. Namely, over the open dense subset of $\mathcal{N}$ one can parameterize the holomorphic bundle by choosing a set of $g$ twists: elements of the complex group $G$, assigned to the $A$-cycles of $\Sigma$. More precisely, let us fix the representatives $a_k$, $k = 1, \ldots, g$, of the $A$-cycles and let $\tilde{\Sigma}$ be the surface $\Sigma$ with the small neighborhoods of $a_k$ removed. Topologically $\tilde{\Sigma}$ is a sphere with $2g$ holes.

118
The boundary of the neighborhood of \( a_k \) consists of two circles \( a_k^\pm \). In order to glue back the surface \( \Sigma \) one has to attach the projective transformations \( \gamma_k \), which map \( a_k^+ \to a_k^- \). These transformations generate Shottky group. On the sphere one can find such a gauge transformation \( h \), that

\[
d'^A = h^{-1}\bar{\partial}h
\]

Obviously,

\[
h(g_k(x))|_{a_k^-} = H_k(x)h(x)|_{a_k^+}
\]

where \( H_k \) is a holomorphic \( G \)-valued function, defined in the vicinity of \( a_k^\pm \). Generically one can find constant representative of \( H_k \) (this is a Riemann-Hilbert problem).

Once such a gauge \( h \) transformation is chosen, the equation for \( \phi \) can be restated in words as the following: find a meromorphic form on \( \hat{\Sigma} \), which satisfy the following requirements:

- in the vicinity of \( x_\ell \): \( \phi \sim \frac{P_\ell(x)}{x-x_\ell} \)
- twisting: \( \phi(\gamma_k(x))d\gamma_k(x) = \text{Ad} H_k \phi(x) \, dx \)

The answer can be conveniently written in terms of the Poincaré series \([99]\); let \( x, x_0 \in \mathbb{P}^1 \), introduce

\[
\omega_k[x_0] \in \Omega^1(\mathbb{P}^1) \otimes \text{End}(\mathfrak{g}),
\]

\[
\theta[x, x_0] \in \Omega^1(\mathbb{P}^1) \otimes \text{End}(\mathfrak{g}),
\]

\[
\omega_k[x_0]_zdz = \sum_{\gamma \in \Gamma} \text{Ad}(H_{\gamma}^{-1})d\log \frac{\gamma(z) - \gamma_k(x_0)}{\gamma(z) - x_0} \tag{7.15}
\]

\[
\theta[x, x_0]_zdz = \sum_{\gamma \in \Gamma} \text{Ad}(H_{\gamma}^{-1})d\log \frac{\gamma(z) - x}{\gamma(z) - x_0}
\]

where the sum runs over all elements of the Shottky group. The latter is a free group with \( g \) generators - loops around \( B \)-cycles. One assigns to the "word" \( \gamma = b_{i_1}^{n_1}b_{i_2}^{n_2} \ldots b_{i_k}^{n_k} \) an element of \( G \): \( H_\gamma = H_{i_1}^{n_1}H_{i_2}^{n_2} \ldots H_{i_k}^{n_k} \). In (7.15) \( x_0 \) is an auxiliary point on the sphere. The solution has the form:

\[
\phi[x_0] = \sum_{k=1}^{g} \omega_k(w_k)[x_0] + \sum_{i=1}^{L} \theta(p_i)[x_i, x_0] \tag{7.16}
\]

The momenta \( w_k \) are defined from the condition:

\[
\int_{a_k} \phi = w_k
\]

119
The final remark concerns the vanishing of the residue at the point $x_0$:

$$
\sum_k Ad(H_k^{-1})(w_k) - w_k + \sum_i p_i = 0
$$

(7.17)

This equation we met in the degenerate form in the section §4.1. It has the meaning of the moment map for the action of $G$ on the product of $g$ copies of $T^*G$ and the coadjoint orbits of $p_k$.

An easy computation shows, that the reduced symplectic form is the result of the Hamiltonian reduction with respect to the natural action of the group $G$ with moment (7.17) of the sum of the symplectic forms on the orbits, attached to the points $x_i$ and of the $g$ copies of the Liouville form on the $T^*G$, where the momentum for $H_k$ is $w_k$.

7.1.5. Action-angle variables

We first recall the construction of Hitchin in the case of the compact curve of genus $g > 2$.

Given a point in the moduli space of Higgs bundles one can construct a spectral curve $S \subset \mathbb{P}(T^*\Sigma \oplus 1)$:

$$\mathcal{R}(x, \lambda) = \det (\phi(x) - \lambda)$$

where $\lambda$ is a linear coordinate on the fiber of the cotangent bundle $T^*\Sigma$. This curve is well-defined, since the equation, which defines it, is gauge invariant.

The curve $S$ is $N$-sheeted ramified covering of $\Sigma$, its genus can be computed by the adjunction formula or using Riemann-Hurwitz theorem.

$$g(S) = N^2(g - 1) + 1$$

which agrees with the dimension of the moduli space of stable bundles over $\Sigma$. Denote by $p$ the projection $S \to \Sigma$.

Given a stable bundle $\mathcal{E}$ over $\Sigma$ we can pull it back onto $S$. There is a line subbundle $\mathcal{L} \subset p^*\mathcal{E}$, whose fiber at generic point $(x, \lambda)$ is an eigenspace of $\phi(x)$ with the eigenvalue $\lambda$. Conversely, given a line bundle $\mathcal{L}$ on $S$, on can take its direct image, which (again at generic point) is defined as

$$\mathcal{E}_x = \oplus_{y \in p^{-1}(x)} \mathcal{L}_y$$

Therefore, under the flow, generated by the Hitchin Hamiltonians, the $\mathcal{L}$ changes. It can be shown, that these flows extend to the linear commuting vector fields on the Jacobian $\text{Jac}(S)$ of $S$.  

120
The origin of the Jacobian of the spectral curve can be understood from the following
naive consideration: one can generically diagonalize $\phi$:

$$\phi(z) = \text{diag}(\varphi_1(z), \ldots, \varphi_N(z))$$

where the order of the eigenvalues is not well-defined and changes as $z \in \Sigma$ -
branching points goes around a non-contractible loop. The integrand in the symplectic
form $\text{Tr}\delta\phi\delta\bar{A}$ can be rewritten (locally) as $\sum_i \delta\varphi_i(z)\delta\bar{A}_{ii}$. The moment map equation
$\bar{\partial}A$ = 0 implies, that $\partial\varphi_i = 0$, $\bar{A}_{ij} = 0$, for $i \neq j$. The label $i$ is not well-defined on $\Sigma$,
but on $S$ one can rewrite the integral over domain $U \subset \Sigma$ of a sum over $i$ as an integral
over a union $V \subset S$ of preimages $\pi^{-1}(U)$ of $U$:

$$\int_U \sum_i \delta\varphi_i(z)\delta\bar{A}_{ii} = \int_V \delta\lambda\delta\bar{a}$$

where $(0,1)$ form $\bar{a}$ on $S$ is defined as:

$$\bar{a}(\lambda, z) = \bar{A}_{ii}(z), \quad \varphi_i = \lambda$$

Choose a basis of $A_a$ and $B_b$ cycles in $H_1(S, \mathbb{Z})$. One can find a basis $\omega_a$ in $H^{1,0}(S, \mathbb{C})$, such that

$$\int_{A_a} \omega_b = \delta_b^a, \quad \int_{B_a} \omega_b = \tau_{ab}$$

Neglecting the behaviour near the branching points (which eventually leads to the correct
answer anyway) $\bar{a}$ defines a line bundle $\mathcal{L}$ over $S$ (which is the same bundle $\mathcal{L}$ we discussed
before) and the gauge equivalent ($\bar{a} \sim \bar{a} + \partial \chi$) $\bar{a}$'s define isomorphic $\mathcal{L}$. One can choose a
gauge $\partial\bar{a} = 0$ and rewrite (7.18)as

$$\Omega = \int_S \delta\lambda\delta\bar{a} = \sum_{a=1}^{g(S)} \delta L_a \delta P^a$$

(7.19)

where $L_a = \int_{A_a} \lambda$, $P^a = \left((3\pi)^{-1}\right)^{ab} \int_{B_b} \bar{a} - \tau_{bc} \int_{A_c} \bar{a}$. Thus, the linear coordinates on $Jac(S)$ (which are naturally identified with $P^a$) are the coordinates of the angle-type,
whereas the integrals of $\lambda$ over the corresponding $A$-cycles in $S$ give the action variables.

The construction of the covering spectral curve and abelianization of the problem resembles
Knizhnik’s idea [107] of replacing the correlators of the analytic fields on the covering of the
Riemann surface by the correlators on the underlying Riemann surface with the insertions
of additional vertex operators. Let us also remark, that quite analogous construction was invented by Krichever in [108] in connection to the elliptic Calogero-Moser System.

Degeneration of the spectral curve We shall adopt the same definition of the spectral curve in the case of degenerate \( \Sigma \).

Obviously, the normalization of \( S \) can be also decomposed as the disjoint union of the components \( S_\alpha \), labeled as the components \( \Sigma_\alpha \) and \( S_\alpha \) covers \( \Sigma_\alpha \) with some fixed branching at the points \( x^i_\alpha \). Indeed, the behavior of \( \phi_\alpha \) near the point \( x^i_\alpha \) is known, since the residue is known. Let us fix the conjugacy class of \( p^{ij}_{\alpha\beta} \). Suppose, that it has \( k^1_i \) eigenvalues of multiplicity 1, \( k^2_i \) eigenvalues of multiplicity 2, and so on. Since near the point \( x^i_\alpha \) \( \phi_\alpha \) behaves like:

\[
\phi_\alpha(x) \sim \frac{p^{ij}_{\alpha\beta}}{x-x^i_\alpha}
\]

for appropriate \( j \) and \( \beta \), then \( \lambda \) behaves like

\[
\lambda_m(x) \sim \frac{p_m}{x-x^i_\alpha}
\]

where \( p_m \) is the \( m \)'th eigenvalue of \( p^{ij}_{\alpha\beta} \). Following [109], we can find

\[
N^{ij}_{\alpha\beta} = \sum_{m=1}^{\infty} k^i_m
\]

points \( P_1, \ldots, P_{N^{ij}_{\alpha\beta}} \) above \( x^i_\alpha \), such that the local parameters \( Z_1, \ldots, Z_{N^{ij}_{\alpha\beta}} \) near them are defined:

\[
Z_m = \lambda_m(x) - \frac{p_m}{x-x^i_\alpha}
\]

The discriminant \( \Delta_\alpha \) of \( \phi_\alpha \) is a meromorphic \( N(N-1) \)-differential on \( \Sigma_\alpha \). At each point \( x^i_\alpha \) it has a pole of the order

\[
\phi^i_\alpha = N^2 - \sum_m k^i_m m^2.
\]

The zeroes of \( \Delta_\alpha \) determine the branching points of the covering

\[
S_\alpha \to \Sigma_\alpha
\]

The number of the branching points equals, therefore, to

\[
2N(N-1)(g(\Sigma_\alpha) - 1) + \sum_i \phi^i_\alpha
\]

122
The genus of $S_\alpha$ can be computed with the help of Riemann-Hurwitz formula, which gives:

$$g(S_\alpha) = 1 + N^2(g - 1) + \frac{1}{2} \sum_i \alpha_i^*$$

Now the Hamiltonian flow due to our Hamiltonians produces a motion of the line bundle over $S_\alpha$ and it covers the Jacobian of the completed curve $\overline{S}_\alpha$, therefore, the coordinates of the particles will be determined by the same equation:

$$\theta(\sum_i U^i t_i + Z_0)$$

as in the simplest one-punctured case. Here $\theta$ is a $\theta$-function on the Jacobian of $\overline{S}_\alpha$, and $U^i$ defines an embedding of the moduli space of the holomorphic bundles over $\overline{S}_\alpha$ into the Jacobian, as we have described it.

**7.2. Four dimensional models: ALE spaces and instantons**

The ALE (asymptotically locally euclidean) spaces are the non-compact gravitational instantons. They can be obtained by blowing up and deforming the singularity of the orbifold $\mathbb{C}^2/\Gamma$ where $\Gamma$ is one of the finite subgroups of $SU(2)$ (there is an ADE classification of them). The ALE manifolds have a hyperkahler structure and can be obtained as a hyperkahler quotients of the linear spaces. This is a result due to Kronheimer [110]. Quite amusingly, the moduli spaces of instantons (for the gauge group $U(N)$) on ALE spaces (which have hyperkahler structure on general grounds) can also be described as hyperkahler quontients [111].

The construction of [111]generalizes the ADHM construction. The dimensions of the spaces $V_i, W_i$ encode the topology of the instanton (its behavior at infinity and first and second Chern classes).

The metric on $X$ can be represented in the Eguchi-Hanson form:

$$ds^2 = V^{-1}(dt + \omega \cdot d\vec{x})^2 + V(d\vec{x})^2$$

where $rot\omega = \nabla V$,

$$V = \sum_{i=0}^{n-1} \frac{1}{||\vec{x} - \vec{x}_i||}$$

$\vec{x}$ - three-dimensional vector, and $t$ - "angular" variable (imaginary time).

According to [111], the generalization of ADHM description is:
a set of vector spaces $V_i, W_i, i = 1, \ldots, N, N + 1 \sim 1$;
a set of operators $B_{i,i+1} : V_{i+1} \to V_i, B_{i+1,i} : V_i \to V_{i+1}$;
and $I_i : V_i \to W_i, J_i : W_i \to V_i$

For the gauge group $U(k)$ the spaces $W_i$ have the following sense:

$$\mathfrak{C}^k = \oplus_i W_i \otimes R_i$$

where $R_i$ - irreducibles of $\Gamma$ (for $\Gamma = \mathbb{Z}_N$ they are all one-dimensional).

The holomorphic symplectic form is given by:

$$\Omega = \sum_i \text{Tr}(\delta B_{i,i+1} \wedge \delta B_{i+1,i} + \delta J_i \wedge \delta I_i)$$

Introduce the ”monodromies”

$$\mathcal{B}_i = \prod_{k=0}^N B_{i+k,i+k+1} : V_i \to V_i$$

(the product is naturally ordered) and consider the $\text{End}(W_i)$-valued functions:

$$\mathbf{F}_i(t) = I_i \frac{1}{t - \mathcal{B}_i} J_i$$

It is clear that they are invariant under the action of $GL(V_i)$. The following commutation relation hold:

$$\{ \mathbf{F}_i(t_1)^A, \mathbf{F}_j(t_2)^B \} = f_{ij}^{AB} \frac{\mathbf{F}_i(t_1)^C - \mathbf{F}_i(t_2)^C}{t_1 - t_2}$$

(7.20)

where $A, B, C$ denote the matrix indices, and $f_{ij}^{AB}$ are the structure constants of the Lie algebra $\text{End}(W_i)$.

The algebra (7.20) is isomorphic (for any $i$) to the algebra of the Poisson brackets on the gauge invariant functionals on a sphere [112]. Its physical interpretation is related to the exceptional divisors - $\mathbb{C}P^1$‘s which are glued to the ALE space when the singularity is blown up.

It is possible to rewrite the algebra (7.20), by introducing the $\text{End}(\mathfrak{C}^k)$-valued function of two variables: $t$ and $q$.

$$\mathbf{F}(t, q) = \sum_i q^i p_i(\mathbf{F}_i(t))$$

(7.21)

where $p_i$ is the embedding of $\text{End}(W_i)$ into $\text{End}(\mathfrak{C}^k)$. We get:

$$\{ \mathbf{F}(t_1, q_1)^A, \mathbf{F}(t_2, q_2)^B \} = f_{ij}^{AB} \frac{\mathbf{F}(t_1, q_1 q_2)^C - \mathbf{F}(t_2, q_1 q_2)^C}{t_1 - t_2}$$

(7.22)
The dimension of the moduli space equals

$$
\sum_{i=1}^{N} \{2v_i w_i - (v_i - v_{i+1})^2\} \quad (7.23)
$$

where $v_i = \dim V_i$, $w_i = \dim W_i$. It follows from (7.20) that the traces of the powers of the residues of $\mathbb{F}_i(t)$ commute. Also, the positions of the poles commute as well. Since

$$
B_{i,i+1} B_{i+1} = B_i B_{i,i+1} \quad (7.24)
$$

the poles (without multiplicities) of $\mathbb{F}_i(t)$ for different $i$ are all the same.

The Hamiltonian system can be pushed down to the reduced phase space since the commuting Hamiltonians we found are invariant under the group action. The full set of integrals of motion is given by

$$
I_{m,n,i} = Tr W_i (Res_{t=m} \mathbb{F}_i(t))^n \quad (7.25)
$$

For $V_i = \mathbb{C}, W_i = 0$ we get ALE space $X$ as the reduced phase space. One can realize it (as a complex manifold) as a surface in $\mathbb{C}^3$, defined by the equation

$$YZ = P(X)$$

with $P(X)$ being the polynomial of the $N$'th degree. The system becomes just the $\mathbb{C}^*$-action:

$$Y \to e^t Y, Z \to e^{-t} Z$$

7.3. Twistor transform

7.3.1. Twistor transform and Classical Integrability

We explain how one can solve equations of motion in $WZW_4$ theory using the twistor transform of the self-dual Yang-Mills equations.

$\mathbb{R}^4$ can be endowed with complex structures parametrized by $\mathbb{P}^1$. Choosing a basepoint complex structure $z^1, z^2$ the others are defined by: $z^A_0 = z^A + u e^{AB} z^B$ and $u$ labels a point $u \in \hat{\mathbb{C}} \cong \mathbb{P}^1 \cong SO(4)/U(2)$.

Given a region $\mathcal{R} \subset \mathbb{R}^4$ its twistor space $\hat{\mathcal{R}} \to \mathcal{R}$ has as fiber the sphere $\mathbb{P}^1$ of complex structures compatible with an orientation. The twistor transform defines a correspondence between solutions of the Yang equation $\mathcal{Y}(\mathcal{R})$ and twistor data $\mathcal{T}(\hat{\mathcal{R}})$ defined by:

$$\mathcal{Y}(\mathcal{R}) \equiv \{ g : \omega \wedge \bar{\partial}(g^{-1} \partial g) = 0 \text{ on } \mathcal{R} \}$$

$$\mathcal{T}(\hat{\mathcal{R}}) \equiv \{ G(s^1, s^2, u) : \hat{\mathcal{R}}_+ \cap \hat{\mathcal{R}}_- \to GL(n, \mathbb{C}) \} \quad (7.26)$$

125
where $\hat{\mathcal{R}}_{\pm}$ are patches defined by the north/south pole and:

1. $G(z_{1}^{u}, z_{2}^{u}, u) = H_{-}^{-1}(x, u)H_{+}(x, u)$ for $0 < |u| < \infty$.

2. $H_{\pm}(x, u)$ holomorphic in $u$ for $|u| < \infty, |u| > 0$

Briefly, choosing a gauge $A^{(1,0)} = 0, A^{(0,1)} = -\partial gg^{-1}$ the SDYM are satisfied iff \( \forall u, F^{(0,2)} = 0 \) which holds iff \( \partial_{A}^{(0,1)}u H(x, u) = 0 \). Imposing holomorphy of $H(x, u)$ in $u$ forces us to choose two functions $H_{\pm}$ holomorphic on the patches $\hat{\mathcal{R}}_{\pm}$ and related as in item 1 above. We then identify $g = H_{+}(x, u = 0)$ as a solution to the Yang equation (this construction is analogous to the one in [113]).

Note that all solutions to the Yang equation could be obtained by taking different holomorphic functions $G$ and solving the Riemann-Hilbert problem (1.) In this sense the classical equations of motion for a complex group $G$ are integrable.

The twistor representation of the self-duality equations has been extensively studied and used in the search for hidden symmetries [114]. It is also worth noting that the twistor transform gives a generalization to 4D of the important property of holomorphic factorization in $\text{WZW}_2$ [115].
8. Further directions and conclusions

In the conclusions we briefly summarize the results of the thesis and sketch a few directions of further study.

8.1. Results

1. Four-dimensional generalizations of two dimensional conformal theories are proposed.
2. The theories are investigated and, in particular, infinite dimensional current algebras are found.
3. The corresponding symmetry group is constructed in arbitrary number of dimensions.
4. The concept of algebraic sector is formulated. Some algebraic correlators are computed.
5. The free field representation for algebraic sector is found. Conversely, the exact effective actions for the twisted chiral $N = 1$ multiplets are computed in the special gauge background.
6. The anomalous origin of $N = 2$ string target space theories is uncovered.
7. Four-dimensional analogues of Liouville theory are discussed.
8. Four-dimensional holomorphic blocks are described and the analogue of Verlinde formula for $K3$ is presented.
9. Integrable systems of Hitchin type and their four dimensional counterparts are constructed.

8.2. Further directions

Current algebras and anomalies are extensively used in QCD [17], [116], [18], [19]. It would be interesting to realize whether some of our results could be applied there (see the recent papers [117]).

The local theory of the holomorphic theory can be used in order to get the representations of the current group in high dimensions.

The topological part of the theory has to do with the compactification of the moduli space of instantons, and might be studied combinatorically.

The gravitational part of the holomorphic theories requires further investigations. It seems [35] that one can generalize the results of [118], [119] to the higher dimensions and provide a theory of extended objects, carrying the complex structure.

There are many more subjects to explore ahead of us.
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