About these notes

These notes were prepared as a support for a talk at the graduate workshop "Supersymmetric Field Theories and Their Mathematical Implications" held in March 2012 at the Simons Center in Stony Brook. They are essentially cut out of [1], from which nearly all the material is taken, with a little rearrangement. To the reader looking for a richer exposition we definitely suggest the excellent sources [1,2]. The author warmly thanks Pablo Solis for his careful proofreading and valuable comments.

1 Lagrangian Mechanics

The role of classical mechanics in physics is to provide a framework for describing the evolution of a system in time. From the physicist’s viewpoint, the power of this approach lies in the universality of its fundamental principle: among all the possible evolutions of a system, only those which minimize the action can take place. With the specification of an initial condition, the principle of least action determines the unique fate of the system.

1.1 Mechanics of a point particle

Let us begin with the simplest of mechanical systems, the point particle. The only degrees of freedom are those of the motion within some space $X$ in time, therefore, in order to describe the system it is sufficient to introduce a function

$$x(t) : M^1 \rightarrow X$$

\footnote{Actually, there are certain types of systems for which one cannot determine a unique possibility for their evolution.}
where \( M^1 \) is an \( \mathbb{R} \)-torsor, i.e. an affine space modeled on the reals, on which
\( \mathbb{R} \) acts transitively. This transitive action is of course time translation, \( t \) is
an affine coordinate on \( M^1 \). This choice reflects the following properties of
physical time

- there is no distinguished origin for time, we only measure time differences
- measurements do not depend on when they are performed
- measurements should be expressed by a single real number

We additionally ask that \( M^1 \) carry a translation invariant Riemannian metric, which we use to measure time intervals. Notice that, in identifying
\( M^1 \sim \mathbb{R} \) there are two choices for a unit vector, corresponding to the opposite
directions of the "arrow of time".

We take \( X \) to be a smooth manifold, endowed with a complete Riemannian metric. The second important piece of data related to \( X \) is a function
\( V : X \to \mathbb{R} \), the potential energy.

Let \( F \) denote the space of possible trajectories: i.e. \( F = \text{Map}(M^1, X) \) is
an infinite dimensional function space. Clearly this includes elements which are not realized in nature. Instead, we denote by \( \mathcal{M} \) the space of physical
trajectories, which can be characterized by Newton’s second law

\[
m \ddot{x}(t) = -V'(x(t))
\]

an important feature of this expression is that it treats energy, not force, as the fundamental quantity that determines the motion. This approach will turn out to be indeed more general, or at least more easy to deal with for complicated systems. Let us comment briefly on the features of \( \mathcal{M} \): we will always assume it to be a smooth manifold, the isometries of \( M^1 \) act on \( \mathcal{M} \)

\[
(T_s x)(t) = x(t - s), \quad x \in \mathcal{M}
\]

Notice also that this is a symmetry of \([2]\), together with time reversal \( x(t) \to x(-t) \) which also acts on \( \mathcal{M} \). \( \mathcal{M} \) carries also a symplectic structure, which is familiar from hamiltonian mechanics, although not evident so far. If \( X \) is a complete Riemannian manifold, the map

\[
\mathcal{M} \to TX
\]

\[
x \mapsto (x(t_0), \dot{x}(t_0))
\]

\[\text{the reason for this choice is that otherwise the particle may end up outside of our space}\]

\[\text{This expression only makes sense if } X \text{ is } E^1, \text{ not for a general Riemannian manifold, on which we’d need to use the covariant derivative}\]
is a diffeomorphism, for fixed $t_0$. Then, since the Riemannian structure
gives an isomorphism between the vector bundles $TX \cong T^*X$, composing
this with (4) we get a diffeomorphism $\mathcal{M} \to T^*X$ and through this we
identify the symplectic structure on $\mathcal{M}$ with the pullback of the natural
structure on $T^*X$. We will see that the symplectic structure is naturally
encoded in the lagrangian description, to which we move on next.

1.2 Lagrangian formulation

From the lagrangian viewpoint, the manifold $\mathcal{M}$ corresponds to the critical
submanifold of $\mathcal{F}$ for a function

$$S : \mathcal{F} \to \mathbb{R}$$

called the action. In particular, a typical choice is that $\mathcal{M}$ is the submanifold
on which

$$\delta S = 0 \quad \text{Euler-Lagrange equation}$$

where $\delta$ will be introduced below. The dynamics of physical systems is
then formulated in terms of a variational problem, a viewpoint that finds
analogues in geometry. Actually, the action typically evaluates to infinity
on most of $\mathcal{F}$, even inside $\mathcal{M}$, unless $M^1$ is taken to be compact. Therefore
the quantity of interest is rather a density which integrates to $S$ along $M^1$.
The lagrangian density is the fundamental object we are after, formally it
is a map

$$L : \mathcal{F} \to \Omega^0(M^1)$$

where $\Omega^0(M^1)$ denotes densities on $M^1$, or equivalently the space of twisted
1-forms on $M^1$, with twisting by the orientation bundle (we will formalize
this below). In other words, $L$ is of the form $g(t)|dt|$. The lagrangian density
is well defined on the whole $\mathcal{F}$, but integrating it may clearly give infinity.
One should then study integrals of $L$ on finite time intervals, which corre-
sponds to studying how the system evolves within a certain time interval.
An appealing property of $L$ is that one can deduce the Euler Lagrange equa-
tions from it, without the need for $S$, this is the first hint at the notion of
locality, a fundamental concept for field theory, which we will formalize and
use later.

Let us now revisit the problem of the point particle, to get acquainted
with the lagrangian formalism. The lagrangian in question is

$$L(x) = \left[ \frac{m}{2} |\dot{x}(t)|^2 - V(x(t)) \right] |dt|$$
indeed in classical mechanics, the lagrangian is just the difference of kinetic energy and potential energy: note that it’s rather easy to write it down for most systems, as opposed to a description in terms of forces. For any fixed \( x \in F \), the right hand side is precisely a density on \( M^1 \), ready to be integrated to yield the action \( S_{[t,t']} = \int_{t}^{t'} L(x(t)) dt \) over a finite time interval with \( t < t' \). The key point about considering finite time intervals is that they are just as good for deriving the Euler Lagrange equations, with the advantage that \( S \) is finite. To see how this works, consider a “variation of \( x \)” with compact support, say within \( [t_0, t_1] \), then the E.L. equations consist in insisting that \( S_{[t_0,t_1]} \) be stationary to first order. A variation of \( x \) really means a vector of

\[ T_x \mathcal{F} \cong C^\infty(M^1; x^*TX) \]  

(9)
i.e. it is a section of the pullback bundle \( x^*TX \to M^1 \). In practice, consider a small path \( x_u \) in \( F \), parameterized by \( u \in (-\epsilon, \epsilon) \) and such that \( x_{u=0} \equiv x \), then we can Taylor expand

\[ x_u = x + \zeta \cdot u + O(u^2), \quad \zeta \in T_x \mathcal{F} \]  

(10)

Now, without imposing any support condition on \( \zeta \), consider

\[
\frac{d}{du} \bigg|_{u=0} S_{[t_0,t_1]}(x_u) = \frac{d}{du} \bigg|_{u=0} \int_{t_0}^{t_1} L(x_u(t))
\]

\[
= \int_{t_0}^{t_1} \left[ m \langle \dot{x}(t), \nabla_{\dot{x}(t)} \dot{x}(t) \rangle - \langle \text{grad}V(x(t)), \dot{x}(t) \rangle \right] dt
\]

\[
= \int_{t_0}^{t_1} \left[ m \langle \dot{x}(t), \nabla_{\dot{x}(t)} \dot{x}(t) \rangle - \langle \text{grad}V(x(t)), \dot{x}(t) \rangle \right] dt
\]

\[
= \int_{t_0}^{t_1} -\langle m \nabla \dot{x}(t), \dot{x}(t) \rangle + \text{grad}V(x(t), \dot{x}(t)) \right] dt
\]

\[
+ m \langle \dot{x}(t_1), \dot{x}(t_1) \rangle - m \langle \dot{x}(t_0), \dot{x}(t_0) \rangle
\]

(11)

where we used for \( \nabla \) the Levi Civita connection on \( TF \), and \( \langle \cdot, \cdot \rangle \) denotes the inner product induced by the Riemannian metric on \( X \). If we take \( \zeta(t) \) with compact support contained within the integration region, then the last two terms drop. In that case, the remaining integral vanishes for any such \( \zeta \) if and only if

\[ m \nabla_{\dot{x}(t)} \dot{x}(t) + \text{grad}V(x(t)) = 0 \]  

(12)

Therefore the Euler Lagrange equation, a condition on maps between \( M^1 \) and \( X \), coincides with Newton’s law for the point particle.
More generally, if we lift the support condition on \( \zeta(t) \), let \( \delta \) denote the differential on \( M \subset \mathcal{F} \) and define the 1-form \( \gamma_t \in \Omega^1(M) \) \( \\gamma_t(\zeta) = m(\dot{x}(t), \zeta(t)) \)

\[
\delta S_{[t_0, t_1]} = m(\dot{x}(t_1), \zeta(t_1)) - m(\dot{x}(t_0), \zeta(t_0))
= \gamma_{t_1} - \gamma_{t_0}
\]  

(13)

Fix \( t \), then for any \( \zeta(t) \), the \( \gamma_t \) on \( M \) can be thought of as the connection 1-form on a principal real line bundle \( P \to M \). This extends to a 1-paramenter family of such bundles for all \( t \). However, the curvature

\[
\Omega_t := \delta \gamma_t
\]

(14)
is then independent of \( t \), as a consequence of (13). Define \( P_t \) to be the trivial principal \( \mathbb{R} \)-bundle, then equation (13) gives an isomorphism \( P_{t_0} \to P_{t_1} \) consisting in the addition by \( -S_{[t_0, t_1]} \) fiberwise. The consistency of these trivializations relies crucially on the locality of \( S \) in time: let \( t_0 < t_1 < t_2 \), by locality we mean the following “cut and paste property”

\[
S_{[t_0, t_2]} = S_{[t_0, t_1]} + S_{[t_1, t_2]}
\]

(15)
The 2-form \( \Omega \) is closed and nondegenerate, it is a symplectic form on \( M \) and corresponds to the pull-back we described above. Starting from here one can recover the full Hamiltonian description of the mechanics of the system.

We have seen how \( L \) alone determines the Euler-Lagrange equations, the connection one form \( \gamma_t \), and the symplectic form \( \Omega \). Another important piece of information contained in \( L \) are conserved charges, to be intended in the usual hamiltonian sense: observables that Poisson-commute with the Hamiltonian. More concretely we are talking about functions \( Q : \mathcal{F} \to \mathbb{R} \) that satisfy \( \dot{Q} = 0 \) and are associated with infinitesimal symmetries of the system. Thus the lagrangian density nicely encodes all of the information about classical systems. Besides being such a powerful instrument, the lagrangian formulation is also practical for studying more complicated systems, such as rigid bodies, elastic bodies, fields, and more.

2 Lagrangian Field Theory

A field can be thought of as some kind of “function” on a manifold \( M \) which is usually referred to as spacetime. In fact, fields are often regarded sections of some fiber bundle over spacetime. Before we actually start with field theory, let us first spend some words to better qualify what we mean exactly by “spacetime”.

5
2.1 Minkowski spacetime

Since spacetime will play the role that $M^1$ had in mechanics, there are a couple of properties which seem natural to demand. The first is that it be a complete Riemannian manifold, the second is that it carries some positive density that should somehow follow from the metric.

At a first glance, it seems natural to consider spacetime to be the product $M^1 \times X$, this can be naturally endowed with a partial metric\footnote{A metric that specifies inner products of two vectors only if they are both in $TM^1$ or both in $TX$}. This naive choice also has an action by an isometry, time translation, which carries over from mechanics. Overall, this choice is clearly unsatisfactory, both from the mathematical viewpoint: partial metrics do not pull back under diffeomorphisms, and from the physics viewpoint: a Lorentz boost would mix time and space coordinates so that time-translation would be an isometry only in a specific class of frames.

A better description for Minkowski spacetime is as follows: $M^n$ is a real $n$-dimensional affine space, modeled on a vector space $V$ which is endowed with a nondegenerate symmetric bilinear tensor of signature $(1, n-1)$. There is, of course, a $V$-action on $M^n$ which corresponds to translations.

In local coordinates $x^0, x^1, \ldots, x^{n-1}$, where $x^0 = ct$ is time scaled by the speed of light, the metric reads

$$g = dx^0 \otimes dx^0 - dx^1 \otimes dx^1 - \cdots - dx^{n-1} \otimes dx^{n-1} \quad (16)$$

whereas the associated positive densities are

$$|d^n x| = |dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{n-1}|$$
$$|d^{n-1} x| = |dx^1 \wedge \cdots \wedge dx^{n-1}| \quad (17)$$

respectively for $M^n$ and for a generic time slice $x^0 =$constant. A distinguished feature of metrics with this signature is that, as a consequence of the fact that they are not positive definite, there is a whole subspace of $V$, made of vectors with norm zero: the lightcone. If $n > 2$, the lightcone has two components.

Finally, what are the isometries of $M^n$? They are a subgroup of the affine group that certainly contains translations by $V$, since the metric doesn’t depend on any coordinate. Denote $O(V)$ the group of linear transformations that preserve (16), there is a sequence

$$1 \to V \to \text{Iso}(M^n) \to O(V) \to 1 \quad (18)$$
It turns out that, if \( n > 2 \), then \( O(V) \) has four components labeled by two features: 1) whether or not orientation is preserved 2) whether elements of the lightcone are left invariant or exchanged. Clearly, elements in the identity component preserve both the orientation and elements on the lightcone, these are the transformations of main interest.

2.2 A toolkit

Let us spend a few words about some of the objects that we’ll need later, to define a lagrangian field theory. First of all, we would like to formalize the notion of densities that we already used above: a density on a \( n \)-dimensional manifold \( M \) is a tensor field which, in local coordinates, reads \( \ell(x)[d^n x] \). The absolute value ensures that its sign is always specified by \( \ell(x) \), indeed under coordinate transformations it transforms by the absolute value of the Jacobian: a density is a twisted \( n \)-form. Let \( \Omega^{|q|}(M) \) denote the set of densities on a manifold \( M \), furthermore consider the set of \((n-q)\)-forms with the same twisting, and denote them by \( \Omega^{-q}(M) \). Elements of these sets can be thought of as the tensor product of a section of \( \bigwedge^q T M \) and a density. Just as a density can be integrated over the whole manifold, a twisted \( |-q| \)-form can be integrated over a codimension-\( q \) submanifold\(^5\).

The usual Lie derivative and inner product still act on the graded vector space\(^6\) \( \bigwedge^{|q|} \Omega^a(M) \), they are related as usual by the Cartan formula.

As already mentioned, fields are sections of some fiber bundle over space-time \( E \to M \), with fiber a manifold \( X \). To avoid cluttering notation we will restrict attention to maps \( \phi : M \to X \), but everything generalizes to the description of a bundle. Let \( \mathcal{F} = \text{Map}(M, X) \) be the space of smooth maps from \( M \) into \( X \). There is an important distinction between general infinite dimensional manifolds and function spaces: the latter ones are endowed with an evaluation map

\[
e : \mathcal{F} \times M \to X
\]

\[
(\phi, m) \mapsto \phi(m)
\]

Lagrangian field theory is formulated in terms of differential forms on the product space \( \mathcal{F} \times M \), more precisely we use twisted forms on \( M \). So we have a double complex, an element of \( \Omega^{|q|}(M) \) is a \( p \)-form on \( \mathcal{F} \) with values in the space of twisted \( |-q| \) forms on \( M \). We denote by \( \delta \) the exterior derivative

\(^5\)provided its normal bundle is oriented

\(^6\)however, note that twisted forms do not form a ring, but a graded module over untwisted forms
along $\mathcal{F}$, by $d$ that along $M$. We define the total exterior derivative $D$ on $\mathcal{F} + M$ to be

$$D = \delta + d$$

$$D^2 = \delta^2 = d^2 = 0 \quad d\delta = -\delta d$$

$d$ and $\delta$ act on elements of this double complex as denoted in the following diagram

$$
\begin{array}{cccc}
0 & 1 & \ldots & \\
\delta & \rightarrow & \\
\downarrow & & & \\
| & & & \\
\end{array}
$$

As an example, a lagrangian density is a 0-form on $\mathcal{F}$ that maps each $\phi \in \mathcal{F}$ to a density on $M$, so it is an element of $\Omega^{0,0}(\mathcal{F} \times M)$.

The subcomplex of local forms is denoted by $\Omega^{\ast,\ast}_\text{loc}$, where local has the following meaning: let $\alpha \in \Omega^{p,\ast}(\mathcal{F} \times M)$ and $\xi_1, \ldots, \xi_p \in T_\phi \mathcal{F}$, the value of $\alpha$ at $(\phi, m) \in M \times \mathcal{F}$ on $\xi_1, \ldots, \xi_p$ is a twisted form at $m$

$$\alpha_{(\phi, m)}(\xi_1, \ldots, \xi_p)$$

if it depends only on the $k$-jet of $\phi$ at $m$ and on the $\xi_i$, then we say that $\alpha$ is local.

As an example, consider two vector fields on $M$: $\zeta_1, \zeta_2$ and let $X = \mathbb{R}$, then

$$L = \zeta_1 \zeta_2 \phi(m) \phi(m) \left| d^n x \right|$$

is the product of a function and a density on $M$ ($\phi$ is a smooth function on $M$, the $\zeta$’s just take derivatives, giving back a smooth function). $L$ is also local: we can write the function in terms of the evaluation map $e$ which is itself local

$$\zeta_1 \zeta_2 e \ e$$

taking a finite amount derivatives reserve locality, so does multiplication by another local function, therefore $L$ is a local $(0, |0|)$-form on $\mathcal{F} \times M$. Indeed it depends only on the 2-jet of the field $\phi$.

Let us conclude by mentioning a result due to Takens’: For fixed $p > 0$ the complex of local differential $(p, | \ast |)$ forms on $\mathcal{F} \times M$, with exterior derivative $d$, is exact except in the top degree $| \ast | = |0|$. 

8
2.3 Descriptions of physical systems

In order to get acquainted with the machinery we just introduced, let us start by revisiting the point particle. Spacetime really is just $M^1$, while $\mathcal{F} = \text{Map}(M^1, X)$ is the space of paths in a Riemannian manifold $X$. Recall that there is a potential energy function $V : X \to \mathbb{R}$. The lagrangian density is

$$L = \left[ \frac{m}{2} |\dot{x}|^2 - V(x) \right] |dt| \quad \in \Omega^{0,0}(\mathcal{F} \times M) \quad (25)$$

there is a difference from our previous formulation: now $x$ is not an element of $\mathcal{F}$, but the evaluation map $x : \mathcal{F} \times M^1 \to X$. The variational 1-form is now

$$\gamma = m \langle \dot{x}, \delta x \rangle \partial_t \otimes |dt| \quad \in \Omega^{1,-1}(\mathcal{F} \times M) \quad (26)$$

where $\dot{x} = \iota(\partial_t)dx$, also recall that $dx, \delta x$ are 1-forms on $\mathcal{F} \times M$ that take values in the pullback bundle $x^*TX$. Define the total lagrangian

$$\mathcal{L} = L + \gamma \quad (27)$$

we now study its differential $D\mathcal{L}$, there are two pieces: a $(1,0)$ piece and a $(2,|-1|)$ one. Let us choose an orientation for $M^1$ and identify $|dt| = dt$, then we have

$$dx = \dot{x} dt$$

$$\gamma = m \langle \dot{x}, \delta x \rangle \quad (28)$$

The first piece then reads

$$(D\mathcal{L})^{1,0} = \delta L + d\gamma$$

$$= m \langle \delta \nabla \dot{x}, \dot{x} \rangle + dt - dV \circ \delta x \wedge dt$$

$$+ m (d \nabla \dot{x}, \delta x) + m \langle \dot{x}, d \nabla \delta x \rangle \quad (29)$$

notice that, when we first differentiate the evaluation map with either $d$ or $\delta$, we obtain a section of $x^*TX$, therefore hitting with another derivative is done through the pullback of the Levi-Civita connection on $X$, this is what we mean by $d\nabla, \delta \nabla$. Using the fact that the torsion vanishes for such a connection, we get

$$\delta \nabla d = -d \nabla \delta \quad (30)$$

we therefore find a cancellation in the sum, leading to

$$(D\mathcal{L})^{1,0} = -m \langle \nabla \dot{x} + \text{grad} V, \delta x \rangle \wedge dt \quad (31)$$
therefore, requiring that the differential of $L$ vanishes is the same as stating Newton’s second law for the point particle. The variational 1-form is here in the guise of “integration by parts”, taking care of proper cancellations (see comment below).

The second component of the differential of the total lagrangian yields instead

$$\left( DL \right)_{2,-1} = m\langle \delta \nabla \dot{x}, \delta x \rangle$$

this is a $(2, -1)$-form, i.e. a 2-form on $F$ with values in $\Omega^0(M)$, and restricts to the symplectic form on $M$, the set of solutions to Newton’s equation.

**Definition** A [lagrangian field theory](#) on a spacetime $M$ is a lagrangian density $L \in \Omega_{\text{loc}}^{0,0}(F \times M)$ and a variational 1-form $\gamma \in \Omega_{\text{loc}}^{1,-1}(F \times M)$ such that $(DL)^{1,0}$ is linear over functions on $M$.

Linearity here means the following: first notice that $T^\phi F$ is a module over $\Omega^0(M)$, then $\beta \in \Omega^{1,*}(F \times M)$ is linear over functions if at each point on $F \times M$, for any $f \in \Omega^0(M)$ and $\hat{\xi} \in T^\phi F$

$$\beta_{(\phi,m)}(f \hat{\xi}) = f(m)\beta_{(\phi,m)}(\hat{\xi})$$

A rule of thumb is that $\delta x$ is linear over functions, while $\delta \nabla dx$ is not. Notice that in (29) it is precisely $\gamma$ that takes care of canceling such non-linear terms. In other words, when we do integration by parts, we do so in order to isolate terms with $\delta x$, as these are the ones that eventually produce the E-L equations, this is accomplished by $\gamma$ here.

All of this might still seem a little awkward: even if we accept that writing down a lagrangian density $L$ is easy, how do we come up with the right $\gamma$? If only the 1-jets of the fields appear in $L$, then there is a canonical choice for $\gamma$. But if higher derivatives appear instead, there is more than one choice, but since $\gamma \in \Omega^{1,-1}(F \times M)$, Takens’ theorem tells us that any two of them are related by addition of a $d$-exact $(1, -1)$-form on $F \times M$.

Now that we have a definition of a lagrangian field theory, we can state precisely what is $M$: the space of classical solutions is the subset of $\phi \in F$ such that the restriction of $(DL)^{1,0}$ to $\{\phi\} \times M$ vanishes. The Euler-Lagrange equations

$$\delta L + d\gamma = 0$$

hold on $M \times M$. Notice that the definition of a lagrangian field theory involves only local forms, hence the E-L equations are themselves local. Elements of $M$ are often called on-shell, while all the others are said to be off-shell.
Although we have deliberately neglected the hamiltonian formalism so far, it plays an important role when switching to quantum theories. We previously promised that one can recover everything from the lagrangian description, in order to maintain our promise we need to introduce an additional piece of data

**Definition** Let $\mathcal{L} = L + \gamma$ define a lagrangian field theory. The associated local symplectic form is

$\omega := \delta \gamma \in \Omega^{2|1}(\mathcal{F} \times M)$ \hspace{1cm} (35)

As promised, the symplectic form is encoded in the lagrangian formulation. Note that on shell we have $\omega = D\mathcal{L}$, which means that $D\omega = 0$ on $\mathcal{M} \times \mathcal{M}$.

We now illustrate in detail the concepts introduced above with an example: a free scalar field living in two spacetime dimensions. Let $x^0, x^1$ be local coordinates on $\mathcal{M}$, then fix the orientation on $\mathcal{M}$ to be $\{x^0, x^1\}$ so that we can identify twisted forms with forms. Our lagrangian is

$L = \frac{1}{2} d\phi \wedge *d\phi = \frac{1}{2} \left[ \left( \frac{\partial \phi}{\partial x^0} \right)^2 - \left( \frac{\partial \phi}{\partial x^1} \right)^2 \right] dx^0 \wedge dx^1$ \hspace{1cm} (36)

by computing $\delta L$, it is easy to realize that we need to have

$\gamma = \partial_0 \phi \delta \phi \wedge dx^1 + \partial_1 \phi \delta \phi \wedge dx^0$ \hspace{1cm} (37)

with this, the equations of motion follow to be

$(\partial_0^2 - \partial_1^2)\phi = 0$ \hspace{1cm} (38)

this is (a two dimensional, massless, version of) the Klein-Gordon wave equation, or a Laplace equation in a Lorentzian metric. However we think of it, this equation determines the behavior $\phi$ at least locally if we specify initial conditions for $\phi$, $\partial_\mu \phi$.

### 2.4 Symmetries and conserved quantities

Before we give a definition of what we mean by *symmetries* in the context of lagrangian field theories, let us extend the notion of locality to vector fields: $\xi \in T\mathcal{F}$ is said to be *local* if the value $\xi_\phi \in \Omega^0(M, \phi^*TX)$ at a point $m \in M$ depends only on the $k$-jet of $\phi$ at $m$. Moreover, we say that a vector
field $\xi \in T(F \times M)$ is decomposable and local if it is the sum of a local vector field $\xi$ on $F$ with a vector field $\eta$ on $M$. Note that both such vector fields preserve the bigrading on differential forms, and the associated Lie derivatives commute with both $\delta$ and $d$.

**Definition** i) A generalized infinitesimal symmetry of $L$ is a local vector field $\hat{\xi}$ on $F$ satisfying

$$\text{Lie}(\hat{\xi})L = d\alpha_{\hat{\xi}} \quad \text{on } F \times M$$

for some $\alpha_{\hat{\xi}} \in \Omega^{0,|1|}_{\text{loc}}(F \times M)$.

ii) A manifest infinitesimal symmetry is a decomposable and local vector field $\xi$ on $F \times M$ satisfying

$$\text{Lie}(\xi)L = 0 \quad \text{on } F \times M$$

**Remarks**

- We say that a generalized symmetry is nonmanifest when we must choose $\alpha_{\hat{\xi}} \neq 0$: nonmanifest symmetries preserve the lagrangian only up to an exact term, as a consequence there is an indeterminacy by addition of any closed form to $\alpha_{\hat{\xi}}$.

- Notice that in the definition of a generalized infinitesimal symmetry we do not specify what happens to $\text{Lie}(\hat{\xi})\gamma$: it can be shown that there exists a $\beta_{\hat{\xi}} \in \Omega^{1,|-2|}(F \times M)$ such that

$$\text{Lie}(\hat{\xi})\gamma = d(-\delta\beta_{\hat{\xi}}) \quad \text{on } \mathcal{M} \times M$$

holds on shell.

- For a manifest symmetry, in good cases we do not need to worry about $\text{Lie}(\xi)\gamma = 0$, it follows directly from $\text{Lie}(\xi)L = 0$.

- For a generalized infinitesimal symmetry, it follows from $[\text{Lie}(\hat{\xi}), \delta] = 0$ that

$$\text{Lie}(\hat{\xi})\omega = d(-\delta\beta_{\hat{\xi}}) \quad \text{in } \Omega^{2,|-1|}(\mathcal{M} \times M)$$

this is a local version of the condition for $\hat{\xi}$ to be hamiltonian, notice that this is automatic in mechanics where $\dim M = 1$ and the right hand side vanishes. For manifest symmetries, $\text{Lie}(\xi)\omega$ vanishes by definition.
**Definition** Let \( \xi = \hat{\xi} + \eta \) be a decomposable and local vector field on \( \mathcal{F} \times M \) which is a *manifest* infinitesimal symmetry. The associated *Noether current* is
\[
j_\xi := (\iota(\xi)\mathcal{L})^{0,[-1]} = \iota(\hat{\xi})\gamma + \iota(\eta)L
\]
(43)

From the Cartan formula, it follows easily that *on shell*
\[
dj_\xi = 0 \quad \delta j_\xi = -\iota(\hat{\xi})\omega - d\iota(\eta)\gamma
\]
(44)

the first equation says that \( j_\xi \) is a conserved quantity on \( \mathcal{M} \times \mathcal{M} \), while the second is the analogue of
\[
d\mathcal{O} = -\iota(\xi_\mathcal{O})\Omega
\]
(45)
in hamiltonian mechanics, where \( \mathcal{O} \) is the conserved charge, \( \xi_\mathcal{O} \) the associated vector field and \( \Omega \) the symplectic form.

Analogously, for a nonmanifest symmetry we have

**Definition** Let \( \hat{\xi} \) be a local vector field on \( \mathcal{F} \) which is a *generalized* infinitesimal symmetry. The associated *Noether current* is
\[
j_{\hat{\xi}} := \iota(\hat{\xi})\gamma - \alpha_{\hat{\xi}}
\]
(46)

Again, one finds that the current is conserved, along with an expression for its charge: *on shell*
\[
dj_{\hat{\xi}} = 0 \quad \delta j_{\hat{\xi}} = -\iota(\hat{\xi})\omega + d\beta_{\hat{\xi}}
\]
(47)

Therefore, for both types of symmetries we find conserved Noether currents, together with an expression for the related charges. We conclude with some applications.

**Example 1** (*time translation: manifest*) For the point particle there is an \( \mathbb{R} \)-action on \( M^1 \) that we call time translation: for \( t \in M^1, s \in \mathbb{R} \) it reads \( T_s(t) = t + s \). This induces an action on \( \mathcal{F} \times M^1 \), namely
\[
T_s(x, t) = (x \circ T_{-s}, T_s(t)),
\]
(48)

the corresponding vector field \( \xi \) on \( \mathcal{F} \times M^1 \) is obtained by differentiating with respect to the parameter \( s \). It can be described by specifying its components on each “coordinate function” (the evaluation map for \( \mathcal{F} \))
\[
\iota(\xi)dt = 1 \quad \iota(\xi)\delta x = -\dot{x}
\]
(49)
since \( \xi \) depends only on the 1-jet of \( x \) it is clearly local, moreover since \( T_s \) preserves both the evaluation map and the density \( |dt| \), it preserves \( L \), which is expressed in terms of them. It follows that \( \text{Lie}(\xi) L = 0 \), so time translation is a manifest infinitesimal symmetry.

The associated Noether current is

\[
\begin{align*}
\tilde{j}_t &= (\iota(\xi) L)^{0,[-1]} \\
&= \iota(\xi) \left[ \left( \frac{m}{2} |\dot{x}|^2 - V(x) \right) \ dt + m \langle \dot{x}, \delta x \rangle \right] \\
&= -\left( \frac{m}{2} |\dot{x}|^2 + V(x) \right)
\end{align*}
\]

(50)

this is minus the total energy (or Hamiltonian) of the point particle.

**Example 2** (time translation: non-manifest) we reconsider the point particle of example 1, since \( \xi \) is decomposable, let \( \hat{\xi} \) be its component along \( \mathcal{F} \). The action of this symmetry is now expressed by

\[
\iota(\hat{\xi}) \delta x = -\dot{x}
\]

(51)

locality is clearly satisfied. \( \hat{\xi} \) is a nonmanifest symmetry of \( L \): since \( \hat{\xi} = \xi - \partial_t \)
from the previous example, \( \text{Lie}(\xi) L = 0 \) implies

\[
\text{Lie}(\hat{\xi}) L = -\text{Lie}(\eta) L = -d \iota(\partial_t) L = -d \left( \frac{m}{2} |\dot{x}|^2 - V(x) \right)
\]

(52)

we have therefore

\[
\alpha_{\hat{\xi}} = -\left( \frac{m}{2} |\dot{x}|^2 - V(x) \right) + C
\]

(53)

where \( C \) is an arbitrary constant due to the indeterminacy tied with non-manifest symmetries. The Noether current then reads

\[
\tilde{j}_{\hat{\xi}} = \iota(\hat{\xi}) \gamma - \alpha_{\hat{\xi}} = -\left( \frac{m}{2} |\dot{x}|^2 + V(x) \right) - C.
\]

(54)

**Example 3** (linear momentum) we keep considering the point particle, now taking \( X = \mathbb{E}^d \). The symmetry in question is translation in the \( j \)th direction: it is an isometry of \( \mathbb{E}^d \) and since \( L \) only depends on \( V \) and on the metric on \( X \), if the potential is translation invariant along the \( j \)th direction, then this is a manifest symmetry. The infinitesimal symmetry has components

\[
\iota(\xi_j) dt = 0 \quad \iota(\xi_j) \delta x_i = \delta^i_j
\]

(55)

for this to be a manifest symmetry it turns out that \( \partial_j V \) must vanish. The Noether current associated with this symmetry is

\[
\tilde{j}_{\xi_j} = m \dot{x}^j
\]

(56)
corresponding to linear momentum in the $j^{th}$ direction.

**Example 4 (energy of a scalar field)** Consider, in the context of our previous example of a scalar field, translation in the $x^0$ direction. This defines a vector field $\xi$ on $F \times M$ with components

$$\iota(\xi) \delta \phi = -\partial_0 \phi \quad \iota(\xi) dx^0 = 1 \quad \iota(\xi) dx^1 = 0.$$  \hspace{1cm} (57)

which is clearly local and decomposable. A little algebra confirms that $\text{Lie}(\xi)L = 0$ on the whole $F \times M$. The conserved Noether current is

$$j_\xi = -\frac{1}{2} \left[ (\partial_0 \phi)^2 + (\partial_1 \phi)^2 \right] dx^1 - \partial_0 \phi \partial_1 \phi dx^0$$  \hspace{1cm} (58)

note that conservation only holds on-shell. The coefficient of $dx^1$ is (minus) an energy density in the form “kinetic+potential”, upon integration on a time slice for any fixed $x^0$, we obtain (minus) the global energy, and the last term drops out.

It is instructive to reconsider time translations also as a nonmanifest symmetry $\hat{\xi}$ acting only along $F$, with components

$$\iota(\hat{\xi}) \delta \phi = -\partial_0 \phi \quad \iota(\hat{\xi}) dx^0 = 0 \quad \iota(\hat{\xi}) dx^1 = 0.$$  \hspace{1cm} (59)

one then finds

$$\text{Lie}(\hat{\xi})L = d\alpha \quad \text{on } F \times M$$
$$\text{Lie}(\hat{\xi})\gamma = \delta\alpha + d\beta \quad \text{on } M \times M$$  \hspace{1cm} (60)

with

$$\alpha_\xi = -\frac{1}{2} \left[ (\partial_0 \phi)^2 - (\partial_1 \phi)^2 \right] dx^1 + C \in \Omega^0_{\text{loc}}$$
$$\beta_\xi = \partial_1 \phi \delta \phi \in \Omega^1_{\text{loc}}$$  \hspace{1cm} (61)

where $C$ is a $d$-closed $(0,|2-1|)$-form on $F \times M$. The associated Noether current is then

$$j_{\hat{\xi}} = -\frac{1}{2} \left[ (\partial_0 \phi)^2 + (\partial_1 \phi)^2 \right] dx^1 - (\partial_0 \phi \partial_1 \phi) dx^0 + C.$$  \hspace{1cm} (62)

**References**
