Phase Space Fluids
and the
Droplet Bosonization Method

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Fluids...

- Macroscopic description of dense many-particle systems
- Important in many areas (HEP, Gravity, Cond. Mat., Cosmo, Appl. Phys., etc.)
- Mathematically intriguing

Classical (Lagrange-Euler) uid mechanics:

\[ \rho(\mathbf{x}), \mathbf{v}(\mathbf{x}) \]

Start form a collection of particles
Nearby particles have nearby velocities
Particles distributed on a \( D \)-dim surface in the 2D-dim phase space
\( \rho(\mathbf{x}) \) is the `thickness' and \( \mathbf{v}(\mathbf{x}) \) is the `position' of the distribution
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“Classical” (Lagrange-Euler) fluid mechanics: $\rho(\vec{x}), \vec{v}(\vec{x})$

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Phase space fluid mechanics

Most general case: particles distributed in phase space

$\phi^\alpha$: 2D-dim single-particle phase space coordinates

$$\{\phi^\alpha, \phi^\beta\}_{sp} = \theta^{\alpha\beta}(\phi)$$

$\theta^{\alpha\beta}$ is canonical structure (can be chosen Darboux)

- Dense collection of particles: phase space density $\rho(\phi, t)$

Its ‘inherited’ field-theory Poisson structure is

$$\{\rho(\phi), \rho(\phi')\} = \sqrt{\theta(\phi_+)} \theta^{\alpha\beta}(\phi_+) \partial_\alpha \rho(\phi_+) \partial_\beta \delta(\phi_-)$$

with $\phi_+ = (\phi + \phi')/2$, $\phi_- = \phi - \phi'$
Alternatively, for ‘test functions’ $f(\phi)$ and $g(\phi)$ and

$$\rho[f] = \int \frac{d\phi}{\sqrt{\theta}} \rho(\phi)f(\phi)$$

the Poisson brackets become

$$\{\rho[f], \rho[g]\} = \rho[\{f, g\}_{sp}]$$

For non-interacting, identical particles with Hamiltonian $h(\phi)$ we have the fluid Hamiltonian

$$H = \int \frac{d\phi}{\sqrt{\theta}} \rho(\phi)h(\phi)$$

which leads to the EOM for $\rho$

$$\dot{\rho} = \{\rho, H\} = -\{\rho, h\}_{sp} = -\theta^{\alpha\beta} \partial_\beta h \partial_\alpha \rho$$

or

$$\dot{\rho} + \phi^\alpha \partial_\alpha \rho = 0 \quad \text{(Liouville’s theorem)}$$
・Obstruction to Lagrangian formulation: Poisson structure is degenerate

・Similar problem in ordinary classical fluids, dealt with Clebsch parametrization; more general here

Casimirs: for any function of a single variable $f(\cdot)$

$$C[f] = \int \frac{d\phi}{\sqrt{\theta}} f(\rho)$$

・Need to set them to constants, then invert Poisson structure (complicated)...

・Or, need to write Poisson in a ‘decoupled’ way that makes the presence of Casimirs explicit.
The ‘cartographic’ transformation

Similar to a hodographic transformation: exchanges one independent for (the only) one dependent variable. [AP]

- Choose Darboux
- Call \((x, p)\) one of the canonical pairs: \(\{\phi^\alpha\} = \{x, p; \sigma^i\}\)
- Define transformation: \(\rho(x, p, \sigma) \rightarrow r(\lambda; x, \sigma)\)

\[
\rho(x, r(\lambda; x, \sigma), \sigma) = \lambda
\]

It can be shown that:

\[
\{r(\lambda; x, \sigma), r(\lambda'; x', \sigma')\} = \delta(\lambda - \lambda') \left[ \partial_x \delta(x - x') \delta(\sigma - \sigma') + \delta(x - x') \theta^{ij} \partial_i r \partial_j \delta(\sigma - \sigma') \right]
\]

- \(\lambda\) part: trivial
- \(x\) part: ‘current algebra’
- \(\sigma\) part: density algebra (in \(D - 2\) phase space dimensions)
Density decouples in Poisson-commuting $\lambda$-copies

One Casimir for each $\lambda$:

$$c(\lambda) = \int dx \, dp \, d\sigma \, \theta(r - p)$$

Casimirs of $\rho$:

$$C[f] = \int d\phi \, f(\rho) = \int d\lambda \, f(\lambda)c(\lambda)$$

Physical meaning: $r$ parametrizes ‘altitude lines’ of $\rho$, as on a geophysical map

Each $r(\lambda)$ represents an (infintesimally thin) ‘droplet’

Lagrangian formulation can be given in terms of ‘Poisson-Kirillov’ action with an extra trivial parameter $\lambda$
Fermion semiclassical droplet

[Early work by Polchinski; Sakita; others]

- Uncertainty-Exclusion: at most one fermion per phase space volume $h^D$ ($h = 2\pi \hbar$ is Planck’s constant)
- Dense configurations: phase space ‘droplet’ of constant density $\rho_o = h^{-D}$
- Determined by the shape and position of its $2D - 1$ boundary
- Can be parametrized in terms of a ‘boundary field’ $R$ representing any of its coordinates:
  
  $$R(x, \sigma) = p|_{\text{boundary}}$$

Non-interacting fermions: ground state consists of filled single-particle energy states up to Fermi level $E_F$

$$h(x, R_{gs}, \sigma) = E_F$$
Droplet configuration constitutes a Hamiltonian reduction of the full density field [AP].

- Put

\[
\begin{align*}
    r(\lambda, x, \sigma) &= R(x, \sigma) & \text{for } \lambda < \rho_o \\
    &= 0 & \text{for } \lambda > \rho_o
\end{align*}
\]

an easily implementable set of second-class constraints. The boundary field satisfies

\[
\{ R(x, \sigma), R(x', \sigma') \} = \frac{1}{\rho_o} \left[ \delta'(x - x')\delta(\sigma - \sigma') + \delta(x - x')\theta^{ij} \partial_i R \partial_j \delta(\sigma - \sigma') \right]
\]

All Casimirs are neutralized

\[
C[f] = \int dx \, dp \, d\sigma \, f(\rho) = f(\rho_o) \int dx \, dp \, d\sigma \, \theta(R - p)
\]

with only remaining Casimir the total particle number

\[
N = \rho_o \int dx \, dp \, d\sigma \, \theta(R - p)
\]
For non-interacting particles the Hamiltonian is

\[ H = \rho_o \int dx \, dp \, d\sigma \, h \, \theta(R - p) \]

and the EOM for \( R \) obtains as

\[
\dot{R} = \{ R, H \} = -\partial_x h - \partial_p h \partial_x R - \theta \ddot{u} \partial_i R \partial_j h \bigg|_{p=R} \\
= \dot{p} - \dot{x} \partial_x R - \dot{\sigma}^i \partial_i R \bigg|_{p=R} \\
= -\partial_x h(x, R, \sigma) - \theta \ddot{u} \partial_i h(x, R, \sigma) \partial_j R
\]

- Correct EOM as induced from underlying particle motion.
- Non-degenerate Poisson structure
- Can be realized in terms of "Poisson-Kirillov" term
- Spin & internal degrees of freedom can be incorporated by appending a compact phase space of volume \( \sim \hbar \); will not be considered here [Linearized form: Karabali, Nair]
The droplet bosonization method

Semiclassical picture can be used as starting point for a perturbatively exact bosonization of the many-body fermion system

- Quantize the Poisson structure of $R$
- Express quantum operators in terms of $R$

Weyl-ordered quantum density operator can be defined as

$$\hat{\rho}(\phi) = \sum_{a=1}^{N} : \delta(\phi - \phi_a) :_W$$

$\hat{\rho}$ is a universal operator for the many-body system

E.g., for any quantum operator $\hat{A} = \sum_a \hat{A}_a$ we have

$$\hat{A} = \int d\phi A(\phi)_W \hat{\rho}(\phi)$$

$A(\phi)_W$ is the Weyl symbol of $\hat{A}$ (Weyl-ordered $\hat{A}$)
\[ \hat{\rho} \text{ satisfies the quantum commutators} \]

\[ [\hat{\rho}[f], \hat{\rho}[g]] = i\hbar \hat{\rho}[\{f, g\}_*] \]

- The noncommutative Moyal bracket appears
- Also known as the ‘sine algebra’ or ‘w-∞ algebra’
- **Not** simply the Poisson bracket turned into \((i\hbar)\) commutator

**First task:** Quantize \( R \)
- PB of \( R \) contain a **current** part (\( \delta' \) affine part) in \( x \) and a **density** part (\( \theta^i \) part) in the residual \( 2D - 2 \) variables \( \sigma \)
- Quantize by turning the second part into Moyal brackets

\[ [\hat{R}(x, \sigma), \hat{R}(x', \sigma')] = i\hbar \rho_o [\delta'(x_-)\delta(\sigma_-) + \delta(x'_-)] \{R(x, \sigma), \delta(\sigma_-)\}_{*\sigma} \]

The ‘quantum droplet algebra’ [AP]
A sort of Kac-Moody algebra in the residual phase space

The coefficient of the ‘central term’ (affine \( \delta' \) term) must be quantized

Coefficient comes out automatically quantized to \( k = 1 \)

Algebra has \textit{unique irreducible} representation

It faithfully reproduces the \textit{full perturbative} Hilbert space of fermion excitations around (any) ground state.

- A legitimate bosonization of the fermion system in any dimension: Bosonic Hilbert space is \textit{unique, irreducible, full} fermionic Hilbert space [AP; earlier, unrelated work by Luther; Haldane; Houghton, Marston; Khveshchenko; etc.]
Second task: Express $\hat{\rho}$ in terms of $\hat{R}$

- The classical expression

$$\hat{\rho} = \int dx \, dp \, d\sigma \, \theta(\hat{R} - p)$$

is not exact quantum mechanically

- Gives results to leading $1/N$ order but not higher corrections
- Reproduces correctly linear phase space operators
- Answer not fully known in higher dimensions

→ Go to 1+1 dimensions where life is easy!
One dimensional systems

- The residual phase space $\sigma$ vanishes
- $\hat{R}(x)$ is a function of the single spatial variable $x$
- It obeys
  \[
  [\hat{R}(x), \hat{R}(x')] = -i2\pi \delta'(x - x') \quad \text{(we put } \hbar = 1)\]
  or in terms of Fourier modes (take $x$ compact of period $2\pi$)
  \[
  [\hat{R}_n, \hat{R}_m] = n \delta_{n+m}
  \]
- A chiral, abelian current algebra
- Generically there are two commuting boundary fields of opposite chirality (two Fermi levels); concentrate on one
- Hilbert space: Fock space of the infinite tower of oscillators $\hat{R}_{-n}$ acting on the ground state
The quantum density operator

\[ \hat{\rho}(x, p) = \int dp \hat{\rho}(x, k)e^{ikp} \]

can be expressed as [Enciso, AP]

\[ \hat{\rho}(x, k) = : \frac{i \int_{x-k/2}^{x+k/2} \hat{R}(s) ds}{4\pi i \sin \frac{k}{2}} : \]

- \( : \cdot : \) denotes normal ordering in \( \hat{R}_n \), \( \hat{R}_-n \)
- Point-splitting in \( x \) proportional to the momentum in \( p \)
  - reminiscent of noncommutative effects
- The limit \( \hbar \to 0 \) corresponds to \( k \to 0 \) and recovers the semiclassical droplet expression \( \hat{\rho}(x, p) = \frac{1}{2\pi} \theta(\hat{R}(x) - p) \)
As a check, the VEV of $\hat{\rho}$ involves only the zero mode $\hat{R}_0$ which is the Casimir

$$\langle \hat{\rho}(x, p) \rangle_o = \frac{1}{2\pi} \int \frac{e^{ik(R_0-p)}}{4\pi i \sin \frac{k}{2}} dk = \frac{1}{2\pi} \theta(R_0-p) \sum_k \delta(R_0-p-\frac{1}{2}-k)$$

reproducing momentum quantization.

For a momentum-dependent single-particle Hamiltonian of the form $\hat{h} = h(\hat{p})$ that admits the vacuum as its ground state

$$E_o = \int dx \, dp \, h(p) \rho_0(x, p) = \sum_{k=0}^{R_0-\frac{1}{2}} h(k)$$

as expected.
Two applications

[Current work with Karabali]

1. Bosonize any single-particle Hamiltonian

Take $h(p) = p^n$

- $n = 1$: chiral relativistic particles
- $n = 2$: non-relativistic particles; etc.

Hamiltonian is of the form

$$\hat{H}_n = \frac{1}{n + 1} \int dx : \hat{R}^{n+1} : + \text{lower order terms}$$

(exact formula in terms of Bernoulli polynomials available but complicated)

- Leading term is dominant in the large-$N$ limit
- Lower order terms include lower powers of $\hat{R}$ and terms containing $\hat{R}'$
A few cases:

\[ \hat{H}_1 = \frac{1}{2\pi} \int dx : \left( \frac{\hat{R}^2}{2} - \frac{1}{8} \right) : \]

- A quadratic Hamiltonian: relativistic bosonization

\[ \hat{H}_2 = \frac{1}{2\pi} \int dx : \left( \frac{\hat{R}^3}{3} - \frac{\hat{R}}{12} \right) : \]

- The famous cubic non-relativistic fermion Hamiltonian (one chiral sector) up to trivial (Casimir) term [Jevicki; Polchinski; etc.]
- Related also to "Chaplygin gas" [Jackiw, AP]

\[ \hat{H}_3 = \frac{1}{2\pi} \int dx : \left( \frac{(\hat{R}^2 - \frac{1}{4})^2}{4} - \frac{\hat{R}\hat{R}''}{4} \right) : \]

- The first Hamiltonian with nontrivial lower order terms; etc.

EOM of \( \hat{R} \) is of the Hamiltonian, nonlinear, higher derivative form
2. Two-body interactions

Consider (symmetric) two-body potential

\[ U = \sum_{a<b} V(x_a - x_b) \]

Potential energy is (drop hats from operators)

\[ U = \frac{1}{2} \left[ \int dx dy \, V(x - y) \rho(x) \rho(y) - \int dx V(0) \rho(x) \right] \]

or in terms of Fourier modes

\[ U = (2\pi)^2 \sum_{n>0} V_n \rho_{-n} \rho_n + \frac{1}{2} (2\pi)^2 \rho_0^2 - \pi V(0) \rho_0 \]

We need both chiral fields \( R \) and \( \bar{R} \). We obtain

\[ U = V_0 \frac{N(N-1)}{2} + \sum_{n>0} (n-N)V_n + \sum_{n>0} V_n (R_{-n}R_n + \bar{R}_{-n}\bar{R}_n - R_n\bar{R}_n - R_{-n}\bar{R}_{-n}) \]
Linearizing the spectrum of the non-interacting system (excitations close to the Fermi sea) we obtain the full Hamiltonian

\[ H = E_o + \sum_{n>0} (v + V_n)(R_{-n}R_n + \bar{R}_{-n}\bar{R}_n) - \sum_{n>0} V_n(R_n\bar{R}_n + R_{-n}\bar{R}_{-n}) \]

with \( v \) the semiclassical Fermi velocity and the constant term

\[ E_o = E_{ni,o} + \frac{1}{2} V_0 N(N-1) + \sum_{n>0} (n - N)V_n \]

- Couples chiral modes
- Valid only for modes \( n < N \); smooth enough potentials
- For \( n \sim N \): nonperturbative effects in \( 1/N \)
Can be diagonalized with Bogoliubov transformation:

$$a_n = \frac{1}{\sqrt{2}} [\cosh(\theta)(R_n + \bar{R}_n) - \sinh(\theta)(R_{-n} + \bar{R}_{-n})]$$

$$b_n = \frac{1}{\sqrt{2}} [\cosh(\theta)(R_n - \bar{R}_n) + \sinh(\theta)(R_{-n} - \bar{R}_{-n})]$$

with

$$\text{th}2\theta = \frac{V_n}{v + V_n}$$

The Hamiltonian becomes

$$H = E_G + \sqrt{v(v + 2V_n)} \sum_{n>0} (a_{-n}a_n + b_{-n}b_n)$$

with

$$E_G = E_{ni,o} + V_0 \frac{N(N - 1)}{2} + \sum_{n>0} \left[ n \left( \sqrt{v^2 + 2vV_n} - v \right) - NV_n \right]$$
Large-$N$ phase transition

- Transformation breaks down if $V_n < -\frac{V}{2}$
- If any of the $V_n < -\frac{V}{2}$ the Hamiltonian becomes unbounded from below
- A phase transition occurs: system goes from homogeneous state to lumped state

Physical origin: if two-body interactions become too attractive, fermions ‘condense’

- Ground state still translationally invariant, but voids form
- Nonperturbative effect: Fermi sea gets depleted
Open questions

- Other interesting applications (in progress)
- Spin, color degrees of freedom (essentially done)
- Interactions singular at coincidence points (Calogero etc.): nonperturbatively different ground state; collective field formulation leads to solitons [AP; Abanov, Wigman]
- Reduction/relation to ordinary fluid mechanics
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- Higher dimensional density operator?

Thank You