Multipole Expansion
in the
Quantum Hall Effect

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Outline

- Chern-Simons effective action: bulk and edge
- Wen-Zee term: shift and Hall viscosity
- Incompressible fluids and W-infinity symmetry
- 1/B expansion, higher-spin fields, coupling to gravity
- Universal and non-universal effects
Chern-Simons effective action

\[ S[A] = \frac{\nu}{4\pi} \int A \theta A = \frac{\nu}{4\pi} \int \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \]

Laughlin state \( \nu = \frac{1}{p} \)

\[ \rho = \frac{\delta S}{\delta A_0} = \frac{\nu}{2\pi} B = \frac{\nu}{2\pi} (B + \delta B(x)) \]

Density \( J^i = \frac{\delta S}{\delta A_i} = \frac{\nu}{2\pi} \varepsilon^{ij} E^j \)

Hall current

Introduce Wen's hydrodynamic matter field \( a_\mu \) and current \( j^\mu = \varepsilon^{\mu\nu\rho} \partial_\nu a_\rho \)

\[ S[A] = \int \rho_0 A_0 + \int -\frac{\gamma}{2} a \theta a + A \cdot j \sim \int -a \theta a + A \theta A \]

\( \gamma = \frac{2\pi}{\nu} \)

- Hall current is topological
- Sources of \( a_\mu \) field are anyons
- Needs boundary action \( S_b[\varphi] \), \( A|_b = \partial \varphi \) massless edge states
- Bulk topological theory is tantamount to conformal field theory on boundary

universal transport coeff. \( \sigma_H = \frac{\nu}{2\pi} \)
Wen-Zee-Fröhlich action

- Add spatial metric background $g_{ij}$ and coupling to $O(2)$ spin connection $\omega_\mu$
  
  $$g_{ij} = e_i^a e_j^a, \quad \omega_{\mu}^{ab} = \omega_\mu(e)\varepsilon^{ab}, \quad i, j, a, b = 1, 2, \quad \delta g_{ij} = \partial_i u_j + \partial_j u_i$$  

$$S[A, g] = \frac{1}{2\pi} \int -\frac{1}{2\nu} a da + j \cdot (A + s\omega) = \frac{\nu}{4\pi} \int A dA + 2s A d\omega + s^2 \omega d\omega$$

$$\rho = \frac{\delta S}{A_0} = \frac{\nu}{2\pi} \left( B + \frac{s}{2} \mathcal{R} \right)$$  

Wen-Zee shift

$$N = \nu N_\phi + \nu s \chi$$

$$T_{ij} = -2 \frac{\delta S}{\delta g^{ij}} = \frac{\eta_H}{2} \varepsilon_{ik} \dot{g}_{jk} + (i \leftrightarrow j)$$  

Hall viscosity

$$\eta_H = \frac{\rho_0 s}{2}$$

- $\eta_H$ further universal transport coefficient  
  
(Avron et al., Read et al.)

- $s$ intrinsic angular momentum, $s = \frac{p}{2}, \frac{2n-1}{2}$ on resp. Laughlin & n-th Landau L.

- Checks have been done and other quantities have been computed

(Abanov, Gromov et al.; Fradkin et al.; Read et al.; Son et al.; Wiegmann et al.)
Hall viscosity

\[ T_{ij} = \frac{\eta_H}{2} \varepsilon_{(ik} \dot{g}_{j)k} \]

- Constant stirring creates an orthogonal static force, non dissipative
\[ g_{ij} = e_i^a e_j^a, \quad \omega^{ab}_\mu = \omega_\mu(e) \varepsilon^{ab}, \quad i, j, a, b = 1, 2, \quad \delta g_{ij} = \partial_i u_j + \partial_j u_i \quad \text{strain} \]

\[
S[A, g] = \frac{1}{2\pi} \int -\frac{1}{2\nu} \alpha d\alpha + j \cdot (A + s \omega) = \frac{\nu}{4\pi} \int A dA + 2s A d\omega + s^2 \omega d\omega
\]

\[
\rho = \frac{\delta S}{A_0} = \frac{\nu}{2\pi} \left( B + \frac{s}{2} \mathcal{R} \right)
\]

\[
T_{ij} = -2 \frac{\delta S}{\delta g^{ij}} = \frac{\eta_H}{2} \varepsilon_{ik} \dot{g}_{jk} + (i \leftrightarrow j)
\]

- \( S \) is invariant under **time-independent** diffeomorphisms only
- Hall viscosity vanishes for conformal metrics \( g_{jk} = \sqrt{g} \delta_{jk} \)
  
  **time-dep. area-preserving diffeomorphisms**
  
  \( \delta x^i = \varepsilon^{ij} \partial_j w(t, x), \quad \delta g = 0 \)
Another derivation based on the symmetry of incompressible fluids under area-preserving diffs, the W-infinity symmetry.

Hall states have excitations of dipoles (Wen-Zee) and higher multipoles.
Quantum incompressible fluids

• Area-preserving diffeomorphisms of incompressible fluids

\[ \int d^2x \, \rho(x) = N = \rho_o A \quad \Rightarrow \quad A = \text{constant} \]

• Fluctuations of the fluid are described by generators of the symmetry

  \[
  \delta \rho(z, \bar{z}) = \{ \rho, w \}_P \quad \text{Poisson brackets} \quad \delta z = \{ z, w \}_P
  \]

  \[
  \delta \rho(z, \bar{z}) = i\langle \Omega | [\hat{\rho}, \hat{w}] | \Omega \rangle = \{ \rho, w \}_M \quad \text{Moyal} \quad \rho(z, \bar{z}) = \langle \Omega | \hat{\rho} | \Omega \rangle
  \]

  \[ W_\infty \text{ algebra} \quad \text{(in momentum space GMP sin-algebra)} \]

• Generators at the edge \( z = R e^{i\theta} \) are higher-spin currents:

  \[
  W^0 = \psi^\dagger \psi, \quad W^1 = \psi^\dagger \partial_\theta \psi \sim H, \quad W^2 = \psi^\dagger \partial_\theta^2 \psi, \ldots
  \]

• CFT fully developed and matches Jain hierarchy: \( W_\infty \) minimal models

  (A.C., Trugenberger, Zemba '96)
Bulk fluctuations in lowest Landau level are non-local:

\[ \delta \rho(z, \bar{z}) = i \langle \Omega | [\hat{\rho}, \hat{w}] | \Omega \rangle = i \sum_{n=1}^{\infty} \frac{\hbar^n}{B^n n!} \left( \partial^n_{\bar{z}} \rho \partial^n_z w - \partial^n_z w \partial^n_{\bar{z}} \rho \right) \]

(Iso, Karabali, Sakita)

- can be expressed in terms of fields of increasing spin, traceless & symmetric

\[ \delta \rho = \frac{i}{B} \partial_{\bar{z}} (\rho \partial_z w) + \frac{i}{2B^2} \partial^2_{\bar{z}} (\rho \partial^2_z w) + \cdots + \text{h.c.} \]

\[ = i \partial_{\bar{z}} a_z + \frac{i}{B} \partial^2_{\bar{z}} b_{zz} + \cdots + \text{h.c.} \]

- Recover Wen hydrodynamic field \( a_\mu \) plus \( \frac{1}{B} \) correction \( b_{\mu k} \) (\( \mu = 0, 1, 2, \ k = 1, 2 \))

\[ a_\mu = (a_0, a_z, a_{\bar{z}}), \quad b_{\mu k} = (b_{0z}, b_{0\bar{z}}, b_{zz}, b_{\bar{z}\bar{z}}, b_{z\bar{z}}, b_{\bar{z}z}) + \text{gauge symmetry} \]

\[ j^\mu = j_{(1)}^\mu + j_{(2)}^\mu + \cdots, \quad j_{(1)}^\mu = \varepsilon^{\mu \nu \rho} \partial_\nu a_\rho, \quad a_\rho \to a_\rho + \partial_\rho f \]

\[ j_{(2)}^\mu = \frac{1}{B} \varepsilon^{\mu \nu \rho} \partial_\nu \partial_k b_{\rho k}, \quad b_{\rho k} \to b_{\rho k} + \partial_\rho v_k \]

- Expressions determined by current conservation and gauge symmetry
\[ j^\mu = j^{(1)}_\mu + j^{(2)}_\mu + \cdots, \quad j^{(1)}_\mu = \varepsilon^{\mu\nu\rho} \partial_\nu a_\rho, \quad a_\rho \rightarrow a_\rho + \partial_\rho f \]
\[ j^{(2)}_\mu = \frac{1}{B} \varepsilon^{\mu\nu\rho} \partial_\nu \partial_k b_{\rho k}, \quad b_{\rho k} \rightarrow b_{\rho k} + \partial_\rho v_k \]

- \( a_\mu (b_{\mu k}) \) have 1 (2) degrees of freedom
- Assume Chern-Simons dynamics for the \( b_{\mu k} \) field too \( (\text{Gaberdiel et al.}) \)

\[ S[A] = S^{(1)}[A] + S^{(2)}[A] + \cdots \]

\[ S^{(2)}[A] = \int -\frac{1}{B2\gamma} b_k d b_k + A \cdot j^{(2)} = -\frac{\gamma}{2B} \int (\Delta A) dA \quad b_k = b_{\mu k} \ dx^\mu \]

\( O \left( \frac{k^2}{B} \right) \) correction to density and Hall conductivity \( (\text{Hoyos, Son}) \)

Multipole expansion
\[ \delta \rho = \varepsilon^{ij} \partial_i a_j + \frac{1}{B} \varepsilon^{ij} \partial_i \partial_k b_{jk} + \cdots \]

\[ \delta \rho_{\text{charge}} = q \delta(\vec{x}) \quad q = \int d x^i a_i \]
\[ \delta \rho_{\text{dipole}} = \frac{1}{B} p^k \partial_k \delta(\vec{x}) \quad p_k = \int d x^i b_{ik} \]
Coupling to gravity

• Spin-two field allows independent coupling to the metric: the stress tensor is

\[ t^{\mu k} = \varepsilon^{k\ell} \varepsilon^{\mu
u \rho} \partial_{\nu} b_{\rho \ell}, \quad \partial_{\mu} t^{\mu k} = 0, \quad t^{ij} = -\dot{b}_{jk} + O(b_{0n}) \]

• Stress tensor is conserved and symmetric in space indices (Non-Relativistic)

\[ \delta \rho = \varepsilon^{ij} \partial_{i} a_{j}, \quad \delta Q = \int_{D} d^{2}x \delta \rho = \int_{\partial D} dx^{i} a_{i} \quad \text{net charge fluctuation at boundary} \]

\[ \delta P^{k} = \int_{D} d^{2}x t^{0k} = \varepsilon^{k\ell} \int_{\partial D} dx^{i} b_{i\ell} = \varepsilon^{k\ell} u_{\ell} \quad \text{net momentum fluctuation} \]

• Insert metric coupling in the second order action

\[ S(2) [A, g] = \int - \frac{1}{B 2\gamma} b_{k} d b_{k} + A \cdot j_{(2)} + \lambda g_{ij} t^{ij} = \frac{\nu s}{4\pi} \int - \frac{1}{B} \Delta A dA + 2 A d\omega \]

• Obtain: - earlier correction to \( \sigma_{H} \sim B^{-1} \)

- **Wen-Zee action** (quadratic approx) \( S_{WZ} = \frac{\nu s}{2\pi} \int A d\omega \sim B^{0} + B^{1} \)
• composite fermion = dipole of unbalanced charges: ⚫ e, ⚪ h
  charge fluctuation at the boundary

\[
\int \rho(x) \, d^2x = N, \quad \int \frac{x^2}{\ell^2} \rho(x) \, d^2x = \frac{N^2}{2\nu} - N(s - 1) \quad \text{exact sum rule} \quad \nu = \frac{1}{p}, \ s = \frac{p}{2}
\]
\[
\frac{1}{N} \sim \frac{1}{B} \quad \text{correction from } S_{(2)}
Hall viscosity

\[ T_{ij} = \frac{\eta H}{2} \varepsilon_{(ik} \dot{g}_{j)k} \]

- Stirring creates a local ordering of dipoles, inversed layer in the density
Wen-Zee vs. W-infinity coupling

- Wen-Zee interaction is the standard coupling of spin to gravity

\[ S_{ab}^{\mu} = \bar{\psi} \gamma_{ab}^{\mu} \frac{1}{4} [\gamma_a, \gamma_b] \psi, \quad S_{ab}^{\mu} \omega_{\mu}^{abc} \rightarrow J^{\mu} \omega_{\mu}^{12} \quad \text{D=2+1 & Non-Relativistic} \]

- Minimal coupling of electrons is problematic in the lowest Landau Level

\[ P^i = \frac{m}{e} \rightarrow ?? \quad m \rightarrow 0 \quad \text{cf. Generalized Galilean symmetry} \]

- W-infinity provides independent sources for \( P^i \) and \( J^i \) (and . . . )

\[ S_{\text{int.}} = \int A_i J^i(a, b, c, \cdots) + g_{ij} T^{ij}(b, c, \cdots) + \gamma_{ijk} S^{ijk}(c, \cdots) + \cdots \]

- spin equivalent to angular momentum to leading order  (up to technicalities)

- Wen-Zee action is obtained to second order but there are higher terms
Universal and non-universal terms

\[ S[a, b_k, c_{k\ell}] = -\frac{1}{4\pi\nu} \int ada + \frac{1}{sB} b_k db_k + \frac{1}{\alpha B^2} c_{k\ell} dc_{k\ell} + (A, g) \text{ couplings} \]

\[ S[A, g] = \frac{\nu}{4\pi} \int \left( 1 - s \frac{\Delta}{B} + \alpha \frac{\Delta^2}{B^2} \right) AdA + 2s \left( 1 - \beta \frac{\Delta}{B} \right) Ad\omega \]

- **Couplings** \( \nu, s, \alpha \) of Chern-Simons actions are universal by matching to observables of CFT on boundary \( S[a, b_k, c_{k\ell}] + \Delta S[a_k = \partial_k \phi, \ b_{kj} = \partial_k v_j, \ldots] \)

- **However** \( \frac{\Delta}{B}, \frac{\mathcal{R}}{B} \) terms are local corrections and can be altered at will. Only first term in each series is universal.

- **Effective action** is a bookkeeping method for disentangling universal and non-universal transport coefficients & quantities.
Third order (in progress)

- W-infinity deformation suggests the spin-3 field, with 2 physical components
  \[ c_{\mu,k\ell} = (c_{0zz}, c_{0\bar{z}z}, c_{z\bar{z}z}, c_{\bar{z}\bar{z}z}, c_{\bar{z}zz}, c_{zz\bar{z}}) \]

- Corrections to electromagnetic current and stress tensor (not-unique)
  \[ j_{(3)}^\mu = \frac{1}{B^2} \varepsilon^{\mu\nu\rho} \partial_\nu (\partial_k \partial_\ell c_{\rho k\ell}), \quad c_{\rho k\ell} \rightarrow c_{\rho k\ell} + \partial_\rho v_{k\ell} \quad (k\ell)\text{-traceless & symm} \]
  \[ t_{(2)}^{\mu k} = \frac{1}{B} \varepsilon^{kn} \varepsilon^{\mu\nu\rho} \partial_\nu (\partial_\ell c_{\rho n\ell}) \]

- And Chern-Simons dynamics
  \[ S_{(3)} [A, g] = \int -\frac{1}{2\alpha B^2} c_{k\ell} d c_{k\ell} + A \cdot j_{(3)} + \lambda g \cdot t_{(2)} \sim \int \frac{\Delta^2}{B^2} A dA + \frac{\Delta}{B} A d\omega \]

- Derivative corrections, including part of gravit. Wen-Zee term \[ \int \omega d\omega \]

- Universality? Need new coupling to three-index background \[ \gamma_{ijk} \]
Conclusion

- Effective action of quantum Hall states can be derived systematically by $1/B$ expansion.
- Building principle is the $W$-infinity (i.e. GMP) symmetry of quantum incompressible fluids.
- Universal quantities can be identified.
- Many aspects to be fully developed.