Geometry and Large N limits in Laughlin states

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Laughlin state

We would like to consider Laughlin state on a compact Riemann surface $\Sigma$ of genus $g$

\[ \text{Aim: Study Laughlin state(s) on } \Sigma \text{ with arbitrary geometry:} \]
- inhomogeneous magnetic field $B$, with flux $N_\Phi = \frac{1}{2\pi} \int_{\Sigma} B \sqrt{g} d^2 x$
- arbitrary solenoid (Aharonov-Bohm) phases around the cycles $\int_a A = \varphi_a$, $\int_B A = \varphi_b$ (flat connections moduli - $Jac(\Sigma)$)
- metric $g$, and curvature $R$,
- complex structure moduli $J$,
- singularities of different types at points $z_1, \ldots, z_n$. 
Main problem: partition function

The Laughlin states have the general form

\[ \Psi_r = \frac{1}{\sqrt{Z}} F_r(z_1, \ldots, z_N), \quad r = 1, \ldots, r_g, \]

where \( F \) is a holomorphic function. (\( r_g = \beta^g \) [Wen-Niu’90])

The partition function is the normalization constant \( Z = \langle F_r, F_r \rangle_{L^2} \). It is a functional of various geometric parameters

\[ Z = Z[g, J, B, \varphi, \ldots] \]

For example, the partition function for planar Laughlin state is

\[ Z = \int_{\mathbb{C}^N} \prod_{i<j} |z_i - z_j|^{2\beta} e^{-B \sum_i |z_i|^2} \prod_{j=1}^N d^2 z_j \]

Central object in Log-gases (Coulomb gas, random matrix \( \beta \)-ensemble).

Main goal: Determine \( \log Z[g, J, B, \varphi, \ldots] \) in the limit of large number \( N \) of particles.
Why: geometric adiabatic transport

[Thouless et.al.; Avron, Seiler, Simon, Zograf, ...]. Laughlin states on a Riemann surface \((\Sigma, g, J)\) are holomorphic sections of a vector bundle \(E\) over the parameter space \(Y\) \((Y = \text{Jac}(\Sigma) \times \mathcal{M}_g)\), e.g.

\[
\Psi_r(y) = \frac{1}{\sqrt{Z(y, \bar{y})}} F_r(z_1, \ldots, z_N|y)
\]

Let \(d_Y\) be an exterior derivative along the parameter space. Then adiabatic (Berry) connection and curvature are

\[
\mathcal{A} = \langle \Psi, d_Y \Psi \rangle_{L^2}, \quad \mathcal{R} = d_Y \mathcal{A} = -d_Y \bar{d}_Y \log Z.
\]

- Hall conductance \(\sigma_H\) is a Chern number of this vector bundle over \(Y = \text{Jac}(\Sigma)\) (flat connections moduli) [Thouless et.al.'82'85, Tao-Wu'85, Avron-Seiler’85, Avron-Seiler-Zograf’94]
- Anomalous Hall viscosity \(\eta_H\), associated with the adiabatic transport on the moduli space of a torus \(M_1\). IQHE: [Avron-Seiler-Zograf’95, Levay’95], FQHE: [Tokatly-Vignale’07, Read’09]
- Transport on higher genus. IQHE: [Levay’97], FQHE [SK-Wiegmann’15, Bradlyn-Read’15].
Lowest Landau level (LLL) and IQHE state

Consider compact connected Riemann surface \((\Sigma, g, J)\) and positive holomorphic line bundle \((L^k, h^k)\). The latter corresponds to the magnetic field. The curvature \((1, 1)\) form of the hermitian metric \(h^k(z, \bar{z})\) is given by \(F = -i\partial\bar{\partial}\log h^k\). This is the magnetic field strength of total flux \(k\) though the surface \(N_\Phi = \frac{1}{2\pi} \int_\Sigma F = k\). Magnetic field: \(B = g^z\bar{z} F_z\bar{z}\). On the plane and for constant magnetic field \(B = k\), this corresponds to \(h^k_0 = e^{-\frac{B}{2} |z|^2}\). Wave functions can also transform as section of canonical line bundle \(K^s\), i.e. as symmetric tensors with \(s\) holomorphic indices \(\psi_{z\bar{z}...z}(dz)^s\). LLL wave functions

\[
\bar{\partial}_{L^k \otimes K^s} \psi = 0
\]

are holomorphic sections of \(L^k \otimes K^s\),

\[
\psi_i = s_i(z), \quad i = 1, \ldots, N_k = \dim H^0(\Sigma, L^k \otimes K^s) = k + (1 - 2s)(1 - g)
\]

IQHE state: take \(N_k\) points on \(\Sigma\): \(z_1, z_2, \ldots, z_{N_k}\). The (holomorphic part \(F\) of the) IQHE state is Slater determinant:

\[
F(z_1, \ldots, z_{N_k}) = \det[s_i(z_j)]_{i,j=1}^{N_k}
\]
Arbitrary metric and inhomogeneous magnetic field

The advantage of the language of holomorphic line bundles is that it gives us a clear idea how to put the Laughlin state on $\Sigma$ with arbitrary metric $g$ and inhomogeneous magnetic field $B$. Consider some fixed (constant scalar curvature) metric $g_0$, and constant magnetic field $B_0$ (and corresponding hermitian metric $h_0(z, \bar{z})$). Arbitrary metrics are parameterized by:

- Kähler potential $\phi(z, \bar{z})$: $g_{z\bar{z}} = g_0\bar{z}z + \partial_z \bar{\partial}_z \phi$,
- "magnetic" potential $\psi(z, \bar{z})$: $F = F_0 + \bar{\partial} \partial \psi$, $B = g^{z\bar{z}} F_{z\bar{z}}$

$$|\Psi(z_1, \ldots, z_{N_k})|^2 = |\det[s_i(z_j)]|^2 \prod_{j=1}^{N_k} h_0^k(z_j, \bar{z}_j) e^{-k\psi(z_j, \bar{z}_j)} g^{-s}_{z\bar{z}}(z_j, \bar{z}_j)$$
Results: Partition function for IQHE

$$\log Z = \frac{1}{2\pi} \int_{\Sigma} \left[ A_z A_{\bar{z}} + \frac{1-2s}{2} (A_z \omega_{\bar{z}} + \omega_z A_{\bar{z}}) + \left( \frac{(1-2s)^2}{4} - \frac{1}{12} \right) \omega_z \omega_{\bar{z}} \right] d^2z$$

$$+ \mathcal{F}[B, R].$$

$$\mathcal{F}[B, R] = -\frac{1}{2\pi} \int_{\Sigma} \left[ \frac{1}{2} B \log B + \frac{2-3s}{12} R \log B + \frac{1}{24} (\log B) \Delta_g (\log B) \right] \sqrt{g} d^2z$$

$$+ \mathcal{O}(1/B).$$

This holds for surfaces of any genus. Terminology: $A_z, A_{\bar{z}}$ are components of the gauge-connection 1-form for the magnetic field and $\omega_z, \omega_{\bar{z}}$ are components of spin-connection $\omega_z = i \partial_z \log g_{z\bar{z}}$, and $s \in \mathbb{Z}/2$ is gravitational spin.

There is an equivalent local form of expansion of $\log Z$, for density of states (Bergman kernel)

$$\rho_k(z, \bar{z}) = B + \frac{1-2s}{4} R + \frac{1}{4} \Delta B + \frac{2-3s}{24} \Delta(B^{-1}R) + \frac{1}{24} \Delta(B^{-1} \Delta \log B) \ldots$$

[SK’13; SK-Ma-Marinescu-Wiegmann’15]
Integer QHE: relation to Chern-Simons theory

Note that generating functional $\log Z$ and 3d Chern-Simons (Wen-Zee) actions look similar:

$$\log Z = \frac{1}{2\pi} \int_{\Sigma} \left[ A_z A_{\bar{z}} + \frac{1-2s}{2} (A_z \omega_{\bar{z}} + \omega_z A_{\bar{z}}) + \left( \frac{(1-2s)^2}{4} - \frac{1}{12} \right) \omega_z \omega_{\bar{z}} \right] d^2z$$

$$S_{CS} = \frac{1}{4\pi} \int_{\Sigma \times R} \left[ AdA + (1-2s)A d\omega + \left( \frac{(1-2s)^2}{4} - \frac{1}{12} \right) \omega d\omega \right]$$

[Wen-Zee'92, Frohlich-Studer'92, Abanov-Gromov'14]

These two actions are obviously very similar, but one is 2d and another one is in 3d. What is the precise relation?
Relation to Chern-Simons theory

Consider geometric adiabatic transport of IQHE wave function along a contour $C$ in the moduli space $Y = Jac(\Sigma) \times M_g$

Define adiabatic connection:

$$A_Y = \langle \Psi, d_Y \Psi \rangle_{L^2},$$

and adiabatic phase:

$$e^{i \int_C A_Y}.$$

Theorem ([SK-Ma-Marinescu-Wiegmann'15]):

$$\int_C A_Y = \frac{1}{4\pi} \int_{\Sigma_Y \times C} \left[ AdA + (1 - 2s) Ad\omega + \left( \frac{(1-2s)^2}{4} - \frac{1}{12} \right) \omega d\omega \right]$$
Holomorphic sections $s_j(z)$ depend on moduli $y$ and this dependence could be chosen to be holomorphic $s_j(z|y)$. IQHE state $F = \det s_i(z_l)$ is a then holomorphic section $S$ of determinant line bundle $\mathcal{L} = \det H^0(\Sigma, L^k \otimes K^s)$ over the parameter space $Y$. There is a natural Quillen metric on $\mathcal{L}$, given by

$$||S||^2 = \frac{Z}{\det' \Delta_{L^k \otimes K^s}},$$

where $Z$ is the partition function and $\Delta_{L^k \otimes K^s}$ is the magnetic laplacian. Bismut-Freed-Gillet-Soulé formula for the curvature of $\Omega_{\mathcal{L}}$ holds

$$\Omega_{\mathcal{L}} = -2\pi i \int_{X|Y} [Ch(E)Td(X|Y)](4) = -2\pi i \int_{X|Y} d_X CS(A, \omega)$$

where $X|Y$ means the integration goes over the fibers in the fibration $\sigma : X \to Y$, i.e., over $\Sigma_y$ at $y$ fixed. $X$ is ”universal curve”.\s
Definition of Laughlin state (FQHE)

Consider now line bundle \((L^{N_\Phi} \otimes K^s, h^{\beta k})\). But number of points is still
\[ N_k = \frac{1}{\beta} N_\Phi + (1 - g)(1 - 2s/\beta), \]
i.e. only fraction of LLL states is occupied (thus fractional QHE). The (holomorphic part \(F\) of the) Laughlin state satisfies

- \(F(z_1, \ldots, z_{N_k})\) is completely anti-symmetric
- Fix all \(z_j\) except one, say \(z_m\). Then \(F(\cdot, \ldots, \cdot, z_m, \cdot, \ldots, \cdot)\) is a holomorphic section of \(L^{N_\Phi}\).
- Vanishing condition near diagonal \(z_i \sim z_j\) in local complex coordinate system on \(\Sigma\),

\[
F(z_1, \ldots, z_{N_k}) \sim \prod_{i<j} (z_i - z_j)^\beta
\]
Examples

1. Round sphere $S^2$, constant magnetic field: $h_0^k = \frac{1}{(1+|z|^2)^k}$, $N_{\Phi} = \beta k$. Then $s_j = z^{j-1}$, $j = 1, \ldots, k+1$. FQHE state:

$$F(z_1, \ldots, z_{k+1}) = \prod_{i<j} (z_i - z_j)^\beta$$

$$|\Psi(z_1, \ldots, z_{k+1})|^2 = \prod_{i<j} |z_i - z_j|^{2\beta} \prod_{j=1}^{k+1} h^{\beta k}(z_j)$$  \[\text{Haldane'83}\]

2. Flat torus, constant magnetic field:

$s_j = \theta_{\frac{1}{k}, \varphi}(kz + \varphi, \kappa \tau)$, $j = 1, \ldots, k$, $h_0^k = e^{-2\pi i \frac{(z-z')^2}{\tau-\bar{\tau}}}$. FQHE states:

$$F_r(z_1, \ldots, z_k) = \psi \left[ \begin{array}{c} \frac{r}{\beta} \\ 0 \end{array} \right] (\beta z_c + \varphi, \beta \tau) \prod_{i<j} \left( \frac{\psi_1(z_i - z_j, \tau)}{\eta(\tau)} \right)^\beta$$  \[\text{Haldane-Rezayi'85}\]

$$||F_r||^2 = |F_r|^2 \cdot (\sqrt{\text{Im}\tau} |\eta(\tau)|^2)^{\beta k} \prod_{j} h_{0}^{\beta k}(z_j, \bar{z}_j)$$
3. $\Sigma_{g>1}$: $\beta^g$ states. Consider the canonical basis of holomorphic 1-forms $\omega_\alpha$, $\int_{a_\alpha} \omega_\beta = \delta_{\alpha\beta}$, $\int_{b_\alpha} \omega_\beta = \Omega_{\alpha\beta}$.

Abel-Jacoby map: $\Sigma \to \text{Jac}(\Sigma)$, $z \to z_\alpha = \int_{z_0}^{\bar{z}} \omega_\alpha$. Number of particles $N_k = k + 1 - g$.

$$F(z_1, \ldots, z_{N_k}) = \vartheta \left[ \begin{array}{c} \bar{\vec{r}} \\ 0 \end{array} \right] (\beta \bar{z}_c + \varphi, \beta \Omega) \prod_{j<l} E(z_j, z_l), \quad \vec{r} \in [1, \beta]^g$$

where $E(z, w)$ is prime-from (Green function is $G(z, w) = \log E(z, w)$).

Hermitian metric reads

$$\|F\|^2 = |F|^2 \left[ \frac{\det' \Delta}{2\pi \det \text{Im} \Omega} \right]^{-1/2} e^{-\beta \sum_j G_R(z_j)} \prod_j h_0^{\beta k}(z_j) g_{\bar{z}z}(z_j)$$
The proof is based on the free field representation of Laughlin states

$$\sum_{r}^{n_\beta} |\Psi_r|^2 = \int e^{i\sqrt{\beta}X(z_1)} \ldots e^{i\sqrt{\beta}X(z_{N_k})} e^{-\frac{1}{2\pi} S(g,X) \mathcal{D}_g X}$$  [Moore-Read’91]

where sum goes over all degenerate Laughlin states on Riemann surface and the free field action is

$$S = \int_M (\partial X \bar{\partial} X + i \frac{\beta - 2s}{\sqrt{\beta}} X R \sqrt{g} + \frac{i}{\sqrt{\beta}} A \wedge dX)$$

for compactified boson: \( X \sim X + 2\pi \sqrt{\beta} \).

Novelty: “background charge” \( Q = \frac{\beta - 2s}{\sqrt{\beta}} \), gauge connection coupling.
Step 1. The "anomalous part" of the expansion comes from transformation properties under the deformation of the metric and the magnetic field $g_0 \to g = g_0 + \partial_z \partial_{\bar{z}} \phi, \ A_0 \to A = A_0 + \partial \psi,$

$$\int e^{i \sqrt{\beta} X(z_1)} \ldots e^{i \sqrt{\beta} X(z_{Nk})} e^{-\frac{1}{2\pi} S(g,X)} DgX$$

$$= e^{-S_{ano}} e^{-\beta k \sum_j \psi(z_j)} \int e^{i \sqrt{\beta} X(z_1)} \ldots e^{i \sqrt{\beta} X(z_{Nk})} e^{-\frac{1}{2\pi} S(g_0,X)} Dg_0X$$

Step 2. The remainder term $\mathcal{F}[R, B]$ of the expansion of $\log Z$ comes from the interacting path integral

$$\frac{1}{\Gamma(s)} \int_0^\infty d\mu \mu^{s-1} \int e^{-\frac{1}{2\pi} S(g,X)-\mu \int_M e^{i \sqrt{\beta} X(z)} \sqrt{g} d^2 z} DgX ,$$

at $s = -N_k$. 

[Ferrari-SK(JHEP2014)]
\( \log Z \) for Laughlin states

\[
\log Z = \frac{1}{2\pi\beta} \int_{\Sigma} \left[ A_z A_{\bar{z}} + \frac{\beta - 2s}{2} (A_z \omega_{\bar{z}} + \omega_z A_{\bar{z}}) + \left( \frac{(\beta - 2s)^2}{4} - \frac{\beta}{12} \right) \omega_z \omega_{\bar{z}} \right] d^2z
\]

\[
- \frac{1}{2\pi} \int_{\Sigma} \left[ \frac{2 - \beta}{2\beta} B \log B \right] \sqrt{g} d^2z + \ldots
\]

Laughlin states form a non-trivial vector bundle of rank \( \beta^g \) over \( Y = \text{Jac}(\Sigma) \times \mathcal{M}_g \) (In particular, \( \Psi_r \) have non-trivial monodromies under \( S, T \) transformations on torus.)

Yet, for Laughlin states, adiabatic connection and curvature on \( Y = \text{Jac}(\Sigma) \times \mathcal{M}_g \) the are controlled by the partition function, due to the property

\[
\Psi_r(y, \bar{y}) = \frac{F_r(y)}{\sqrt{Z(y, \bar{y})}}.
\]

\[
\mathcal{A}_{rs} = \langle F_r, d_y F_s \rangle_{L^2} = (\partial_y \log Z) \delta_{rs}
\]

\[
\mathcal{R}_{rs} = (id_y d_{\bar{y}} \log Z) \delta_{rs}
\]
New transport coefficient on higher genus

Consider complex structure deformations $g_{z\bar{z}}|dz|^2 \rightarrow g_{z\bar{z}}|dz + \mu d\bar{z}|^2$, where Beltrami differential is $\mu = g_{z\bar{z}}^{-1} \sum_{\kappa=1}^{3g-3} \eta_\kappa \delta y_\kappa$ and $\eta_\kappa$ is a basis of holomorphic quadratic differentials.

Berry curvature, associated with these deformations is

$$\mathcal{R} = id_yd_{\bar{y}}\log Z = 2 \left( \frac{1}{4} \left( 1 - 2 \frac{s}{\beta} \right) \beta k - \frac{c_H}{24} \chi(M) \right) \Omega_{WP},$$

where $\Omega_{WP} = i \int_M d_\bar{y} \mu \wedge d_\bar{y} \bar{\mu} g_{z\bar{z}} d^2z$ is the Weil-Petersson form on the moduli space. Here

$$c_H = 1 - 3\beta \left( 1 - \frac{2s}{\beta} \right)^2$$

is a new transport coefficient, transpiring on higher genus surfaces, since on torus $\chi(M) = 0$.

[SK-Wiegmann'15]
This can be seen from the state as well,

\[ ||F||^2 = |F|^2 \left[ \frac{\det' \Delta}{2\pi \det \text{Im } \Omega} \right]^{-1/2} e^{-\beta \sum_j G_R(z_j)} \prod_j h_0^{\beta k}(z_j) g_{zz}^{-s}(z_j) \]

taking into account that \( g_{zz}^{-1} \delta g_{zz} = d\mu \wedge d\bar{\mu}, \) and
\[ \Omega_{WP} = i \int_M d\bar{y} \mu \wedge d\bar{y} \bar{\mu} \ g_{zz} d^2 z. \]

\[ \mathcal{R} = id_y d_{\bar{y}} \log Z = 2 \left( \frac{1}{4} \left( 1 - \frac{2s}{\beta} \right) \beta N_k - \frac{\chi(M)}{24} \right) \Omega_{WP}, \]

The last term here is the Belavin-Knizhnik holomorphic anomaly for \( \det \) laplacian on Riemann surface.
Singular surfaces

(see also recent work by Laskin-Chiu-Can-Wiegmann, Gromov)

In QHE the singularities of the magnetic field $B \sim \alpha \delta(z)$ are quasi-holes. The singularities of the scalar curvature $R \sim \alpha \delta(z)$ are similar but more involved.

Recall (Cardy-Peschel’88) that the free energy of a 2d system of size $L$ on a surface $\Sigma$ at criticality has the form

$$F = AL^2 + BL - \frac{c\chi(\Sigma)}{6} \log L + O(1).$$

Moreover, for a cone with angle $0 < \alpha < 1$

$$F = AL^2 + BL - \frac{c\chi(\Sigma)}{12} (\alpha + \frac{1}{\alpha}) \log L + O(1).$$
Singular surfaces

Similar formula holds for $\log Z$ for Laughlin states on singular surfaces

$$\log Z = AN_{\Phi}^2 + BN_{\Phi} + C \log N_{\Phi} + O(1),$$

The $O(1)$ term here is very interesting. Recall that in the smooth case it was given by the Liouville action $O(1) = \frac{cH}{12\pi} S_L$. In the cone case the expansion breaks down at this order. In the integer QHE case, we find $C = \zeta(0, \Delta_{\text{cone}})$ and

$$O(1) = \frac{1}{2} \zeta'(0, \Delta_{\text{cone}}) = -\frac{1}{2} \log \det \Delta_{\text{cone}}$$

it is given by the regularized determinant of laplacian on the cone. Spectral theory and zeta functions on singular surfaces were introduced by Cheeger (1979) and studied extensively.
This can be checked by explicit calculation, e.g. for a flat cone, or sphere with two antipodal singularities (american football/spindle)

\[ O(1) = -\frac{1}{2} \log \det \Delta_{\text{cone}} \sim 2\zeta'_2(0, \alpha, 1, 1), \]

where \( \zeta_2 \) is Barnes double zeta function

\[ \zeta_2(s, x, y, z) = \sum_{m,n=0}^{\infty} (mx + ny + z)^{-s} \]

What is the answer for \( O(1) \) term in FQHE (Laughlin state)? This term is "conformal", i.e there is no magnetic scale at this order.
This can be checked by explicit calculation, e.g. for a flat cone, or sphere with two antipodal singularities (American football/spindle)

\[ O(1) = -\frac{1}{2} \log \det \Delta_{\text{cone}} \sim 2\zeta_2'(0, \alpha, 1, 1), \]

where \( \zeta_2 \) is Barnes double zeta function

\[ \zeta_2(s, x, y, z) = \sum_{m,n=0}^{\infty} (mx + ny + z)^{-s} \]

What is the answer for \( O(1) \) term in FQHE (Laughlin state)? This term is "conformal", i.e. there is no magnetic scale at this order. Conjecture: quantum Liouville theory.
Thank you