A model state for filling $1/m$ composite-fermion Fermi-Liquid states in a partially-occupied Landau level with periodic boundary conditions

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- Remarks on emergent versus background quantum geometry
- A modular-invariant formalism on the torus
- A model state for the composite fermion

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• A different vision of quantum Hall geometry: **emergent**, not background:

• so far, formulated on the flat quantum plane

### Classical geometry

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
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<tr>
<td>$\mathbf{x} = x^a \mathbf{e}_a$</td>
<td>Cartesian coordinates</td>
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<tr>
<td>$\mathbf{e}_a \cdot \mathbf{e}<em>b = \delta</em>{ab}$</td>
<td>Euclidean metric</td>
</tr>
<tr>
<td>$A = \frac{1}{2} \epsilon_{ab} dx^a \wedge dx^b$</td>
<td>Area 2-form</td>
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### Quantum geometry

<table>
<thead>
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<tr>
<td>$[R^a, R^b] = -i \epsilon^{ab} \ell_B^2$</td>
<td>Heisenberg algebra</td>
</tr>
<tr>
<td>$\varphi(q, q') = \epsilon^{ab} q_a q'_b \ell_B^2$</td>
<td></td>
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<tr>
<td>$U(q_1)U(q_2) \ldots U(q_n) = \prod_{i&lt;j} e^{\frac{1}{2} \varphi(q_i, q_j)} U(\sum_i q_i)$</td>
<td></td>
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</tbody>
</table>
quantum geometry

- quantum plane is not explicitly endowed with a Laplace-Beltrami operator (does not have intrinsic geometry in a metric sense, though does have a natural (flux) 2-form)

\[ H = \sum_i V_1(R) + \sum_{i<j} V_2(R_i - R_j) \quad [R^a, R^b] = -i \epsilon^{ab} \ell_B^2 \]

\[ V(R^x, R^y) \quad \text{smooth function on } \mathbb{R}^2 \]
\[ \text{analytic function on } \mathbb{R}^2 \]
\[ \text{entire function on } \mathbb{C}^2 \]

\[ V_1(\mathbf{x}) = \int \frac{d^2 q \ell^2}{2\pi} \tilde{V}_c(q) f_n(q) e^{iq \cdot x} \]

- sufficiently rapidly-decreasing to make \( V \) entire
- Fourier transform of (unsmooth) potential
- extremely-rapidly-decreasing Landau orbit form-factor
• clean limit

\[ H_0 = \sum_{i<j} V_2(R_i - R_j) \]

• substrate potentials as perturbation

\[ H = H_0 + \sum_i V_1(R) \]

Look for properties of these clean models
This remarkable “projected Landau level” problem is unique in condensed matter physics:

- It has a short-distance regularization not due to an atomic-scale lattice, but due to the “quantum fuzziness” of its non-commutative geometry.

- This also means that it cannot be described using standard (commutative) quantum field theory.

\[
H_2 = \sum_{i<j} V_2(R_i - R_j) \quad [R^x_i, R^y_j] = -i\delta_{ij} \ell_B^2
\]

particles are identical
The entire “clean limit” problem

Depending on the filling factor $\nu$ and the form of the interaction potential $V_2(\mathbf{r})$, this problem is known to have the following types of ground states:

- incompressible (gapped) translationally-invariant inversion-symmetric topologically-ordered fractional quantum Hall (FQH) states
- compressible (gapless) states with **broken translational symmetry** (stripe and bubble phases, Wigner crystal)
- gapless “Composite Fermi Liquid” (CFL) states with **unbroken translational symmetry** which can be argued to exhibit a neutral fermion Fermi surface

The entire $H_2 = \sum_{i<j} V_2(\mathbf{R}_i - \mathbf{R}_j)$

$[R^x_i, R^y_j] = -i\delta_{ij}\ell^2_B$

exhibits a gapless anomalous Hall effect (AHE)

(like ferromagnetic metals)
In order to clarify what may and may or may not be a fundamental part of the explanation of FQH and CFL let us examine what is and is not present

\[ H_2 = \sum_{i<j} V_2(R_i - R_j) \quad [R_i^x, R_j^y] = -i\delta_{ij} \ell_B^2 \]

• There is no Galilean invariance or Newtonian inertia (standard mass, kinetic energy) left in this model. All dynamics results from the quantum geometry (non-commutativity of $R^x$ and $R^y$)

• This is not a "lowest Landau level" problem. Instead it is an “any Landau level” problem. All information that distinguishes different Landau levels is contained in the form-factor dependence of $V_2$

---

Galilean invariance (also rotational invariance) No

kinetic energy (Galilean or Dirac) No

special “lowest Landau level” physics No
• The key idea for understanding both the Fractional Quantum Hall and Composite Fermi Liquid states is “Flux attachment”

• Flux attachment has an essentially geometric component: it leads to an emergent dynamical metric that induces geometry

• The metric characterizes the shape of the elementary unit of the fluid.
• quantum solid

• unit cell is correlation hole

• defines geometry

measure distances in lattice units

• repulsion of other particles make an attractive potential well strong enough to bind particle

solid melts if well is not strong enough to contain zero-point motion (Helium liquids)
similar story in FQHE:

• “flux attachment” creates correlation hole
• defines an emergent geometry

potential well must be strong enough to bind electron

continuum model, but similar physics to Hubbard model

new physics: Hall viscosity, geometry

but no broken symmetry
A fundamental relation between momentum and electric dipole moment derives from electromagnetism:

\[ \pi_a = \epsilon_{abc} D^b B^c \]

\[ D^a = \epsilon_0 \delta^{ab} E_b + P^a \]

Momentum density

Polarization density

Generator of translations

\[ \bar{p}_a \equiv \left( \frac{\hbar}{\ell_B^2} \right) \epsilon_{ab} R^b = B \epsilon_{ab} (e R^b) \]

2D antisymmetric symbol

\[ [R^a, \bar{p}_b] = i \hbar \delta^a_b \]

(momentum)\textsubscript{a} = B \epsilon_{ab} (electric dipole moment)\textsubscript{b}
flux attachment creates a correlation hole that can bind one or more particles into a composite object

p particles + q “flux” (orbitals)

“flux attachment” Has a shape that defines a metric

displacement of charge relative to center of flux attachment gives an electric dipole

\[ \vec{p}_a = B \epsilon_{ab} (e \delta R^b) \]

guiding center of each particle gets a natural origin, defined by the other particles!

correlation energy \( \rightarrow \) dispersion \( \varepsilon(P, g) \)

“kinetic energy” = electric polarization energy

\[ (velocity^a) = \frac{\partial \varepsilon}{\partial \vec{p}_a} \]
• The key idea is that (at the correct particle density) the Berry phase from motion of the attached vortex cancels the Bohm-Aharonov phase from motion of the charge.

• This means the Lorentz force is canceled by the Magnus force, and the composite object moves in straight lines like a neutral particle.

**Bosons**
- can condense in the $p = 0$ (inversion-symmetric) state with no electric dipole

**Fermions**
- can form a Fermi sea in “momentum” (dipole) space
Berry curvature of the “Flux attachment” of a vortex-like correlation hole modifies the statistics

\[ (-1)^{pq} \xi^p \]

\(-1\) for electrons

\[ = \begin{cases} +1 & \text{composite object is boson} \\ -1 & \text{composite object is fermion} \end{cases} \]

e.g., one electron with \( p = 1, q = 2 \)

\( p \) particles + \( q \) “flux” (orbitals)

- inversion symmetry of FQHE: \( \gcd(p,q) = 1 \) or 2

- exchange phase
the electron excludes other particles from a region containing 3 flux quanta, creating a potential well in which it is bound.

1/3 Laughlin state (composite boson picture)

If the central orbital is filled, the next two are empty

The composite boson has inversion symmetry about its center

It has a “spin” (that couples to Gaussian curvature of its metric)

\[
\begin{array}{ccc}
\frac{1}{2} & \frac{3}{2} & \frac{5}{2} \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\end{array}
\]

\[
L = \frac{1}{2}
\]

\[
L = \frac{3}{2}
\]

\[
s = -1
\]
2/5 hierarchy/Jain state (composite boson picture)

\[
\begin{array}{ccc}
\frac{1}{2} & \frac{3}{2} & \frac{5}{2} \\
1 & 1 & 0 & 0 & 0 \\
\frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5}
\end{array}
\]

\[L = 2\]

\[L = 5\]

\[s = -3\]

\[L = \frac{g_{ab}}{2\ell_B^2} \sum_i R_i^a R_i^b\]

\[Q^{ab} = \int d^2r \, r^a r^b \delta(r) = s\ell_B^2 g^{ab}\]

second moment of neutral composite boson charge distribution
The original model

\[ H_2 = \sum_{i<j} V_2(R_i - R_j) \]

\[ [R_i^x, R_j^y] = -i \delta_{ij} \ell_B^2 \]

\[ \tilde{V}_2(q) \rho(q) \rho(-q) \]

\[ \rho(q) = \sum_{i=1}^{N} e^{i q \cdot R_i} \]

\[ \nu = \frac{N}{N_F} \]

\[ \tilde{V}_2(q) \rho(q) \rho(-q) \]

\[ [\rho(q), \rho(q')] = 2i \sin(\frac{1}{2} q \times q' \ell_B^2) \rho(q + q') \]

The generic symmetry ingredients:

- 2D many-body translation and inversion
  \[ R_i \leftrightarrow a \pm R_i \] (all \( i \))

- Permutation symmetry of the identical particles

- For (spinless) fermions, particle-hole symmetry
  (C) combined with time-reversal (T):

\[ C: \rho(q) \leftrightarrow -\rho(-q) \]

\[ T: i \leftrightarrow -i \]

\[ C^2 = (-1)^{\frac{1}{2} N_F (N_F - 1)} \quad T^2 = 1 \]

Together these preserve the GMP algebra, leave \( H_2 \) unchanged

Total flux = \( N_F \nu = \frac{N}{N_F} \)

Fourier components of guiding-center density

\[ \rho(q) = \sum_{i=1}^{N} e^{i q \cdot R_i} \]
\[ [R_i^x, R_j^y] = -i\delta_{ij}\ell_B^2 \]

- The guiding center algebra is a Heisenberg algebra, with a fundamental representation

\[ [a, a^\dagger] = 1 \]

- states in the one-particle Hilbert space have the holomorphic representation

\[ |\Psi\rangle = f(a^\dagger)|0\rangle \quad a|0\rangle = 0 \]

A holomorphic function

- here

\[ \frac{1}{2\ell_B^2} \tilde{g}_{ab}(R^a - x^a)(R^b - x^b) = \frac{1}{2} (a^\dagger a + aa^\dagger) \]

an arbitrary determinant -1 metric

an arbitrary origin
• The usual “lowest Landau level wavefunction” formalism has

\[ \Psi(x) = f(z) e^{-\frac{1}{4} z^* z / \ell_B^2} \]

holomorphic function

• With a (quasi) periodic boundary condition, this becomes

\[ \psi(z, z^*) \propto \left( \prod_{i=1}^{N\Phi} \tilde{\sigma}(z - w_i) \right) e^{-\frac{1}{4} z^* z / \ell_B^2} \sum_i w_i = 0 \]

modified Weierstrass sigma function*

\[ N\Phi \text{ zeroes} \]

*(slightly modified from Weierstrass’ original definition when the pbc lattice is not square or hexagonal)*

(one for each flux quantum passing through the primitive region of the pbc)
Weierstrass sigma function

\[ \sigma(u) = u \prod_{L \neq 0} \left(1 - \frac{u}{L}\right)e^{\frac{u}{L}} + \frac{1}{2} \frac{u^2}{L^2} \]

\[ \{L\} = \{2m\omega_1 + 2n\omega_2\} \]

both sigma function are “modular invariant” (under changes of basis of the lattice)

• “quasi-modular invariant”

\[ \sum_{L \neq 0} \frac{1}{L^2} \neq \sum_{m \neq 0} \frac{1}{(2m\omega_1)^2} + \sum_{n \neq 0} \left( \sum_{m} \frac{1}{(2m\omega_1 + 2n\omega_2)^2} \right) \]

depends on order in which sum is evaluated (choice of \( \omega_1 \))

\[ G_2(\omega_1,\{L\}) = \tilde{G}_2(\{L\}) + \frac{1}{A} \frac{\omega_1^*}{\omega_1} \]

“almost holomorphic (in \( \tau = \frac{\omega_1}{\omega_2} \)) modular invariant”

vanishes for square/hexagonal lattices

modified sigma function

\[ \tilde{\sigma}(u) = ue^{-\frac{1}{2} \tilde{G}_2 u^2} \prod_{L \neq 0} \left(1 - \frac{u}{L}\right)e^{\frac{u}{L}} + \frac{1}{2} \frac{u^2}{L^2} \]

cancels invariant part of

\[ G_{2k} = \sum_{L \neq 0} \frac{1}{L^{2k}} \]

holomorphic modular invariants ( \( 2k > 2 \))

area of unit cell

\[ A(\{L\}) = \frac{1}{2} i \pi (\omega_1^* \omega_2 - \omega_2^* \omega_1) \]
• uniform states on the torus must be modular invariant (up to topological degeneracy)

• mixed state with density matrix given by trace over topological multiplet must be fully modular invariant.

**use of modified sigma function guarantees modular invariance**
In the Heisenberg-algebra reinterpretation

$$|\Psi\rangle = \prod_{i=1}^{N_\Phi} \sigma(a_i^\dagger - w_i)|0\rangle \quad \sum_i w_i = 0$$

One particle

$$N = 1$$

The filled Landau level is

$$|\Psi\rangle = \left(\prod_{i<j} \sigma(a_i^\dagger - a_j^\dagger)\sigma(\sum_i a_i^\dagger)\right) |0\rangle$$

Filled Level

$$N = N_\Phi$$

The Laughlin states are

$$|\Psi\rangle = \left(\prod_{i<j} \sigma(a_i^\dagger - a_j^\dagger)^m\right) \prod_{k=1}^{m} \sigma(\sum_i a_i^\dagger - w_k)|0\rangle \quad \sum_{k=1}^{m} w_k = 0.$$

$$\nu = \frac{1}{m}$$

Laughlin state

$$N_\Phi = mN$$
Unlike the filled Landau level state, in which the only metric-dependence is the normalization, the Laughlin states depend on the metric choice which fixes the shape of the vortex-like correlation hole around each particle (“attached flux”).

\[ |\Psi\rangle = \left( \prod_{i<j} \sigma(a_i^\dagger - a_j^\dagger)^m \right) \prod_{k=1}^{m} \sigma(\sum_i a_i^\dagger - w_k) |0\rangle \]

\[ \nu = \frac{1}{m} \]

• correlation holes in two states with different metrics

(filled Landau level is a Slater-determinant state with no correlation hole)
• Particle-Hole symmetry (on torus).

The many-fermion states with $\tilde{N}$ holes are described by

\[
\Psi(\{z_i\}) = \Phi_{\tilde{N}+1}(\{z_i\}) \prod_{i<j} \tilde{\sigma}(z_i - z_j | L)
\]

Then applying this to the empty state, just the one-particle state that is the particle-hole conjugate state of the Slater determinant given by

\[
\Phi_{1}(\{z_i\}) = \tilde{\sigma}(\sum_i z_i).
\]

The state of $N$ spinless fermions and $\tilde{N}$ fixed holes $\{\bar{z}_j, j = 1, \tilde{N}\}$ is a Slater determinant given by

\[
\Psi(\{z_i\}, \{\bar{z}_j\}) = \Phi_{\tilde{N}+1}(\{z_i\}; \{\bar{z}_j\}) \prod_{i<j} \tilde{\sigma}(z_i - z_j),
\]

\[
\Phi_{\tilde{N}+1}(\{z_i\}; \{\bar{z}_j\}) = \left( \prod_{j=1}^{\tilde{N}} \tilde{\sigma}(z_i - \bar{z}_j) \right) \tilde{\sigma}(\sum_i z_i + \sum_j \bar{z}_j)
\]
the particle-hole transform

The many-fermion states with $\tilde{N}$ holes are described by

$$\Psi(\{z_i\}) = \Phi_{N+1}(\{z_i\}) \prod_{i<j} \tilde{\sigma}(z_i - z_j)$$

The $\tilde{N}$-particle state that is the particle-hole conjugate state of the $N$-particle state (20) is

$$\tilde{\Psi}(\{z_i\}) = (-1)^{\frac{1}{2} N(N-1)} \tilde{\Phi}_{N+1}(\{z_i\}) \prod_{i<j} \tilde{\sigma}(z_i - z_j)$$

(24)

where

$$\tilde{\phi}_{N+1}(\{z_i\}) = \prod_{i=1}^{N} \int dA_i e^{-\frac{1}{2} \bar{z}_i z_i} \Phi_{N+1}(\bar{z}_j) \Phi_{N+1}(\{z_i\}, \{\bar{z}\}).$$

(25)

can be turned into a finite sum.
Case where composite particle is a fermion

- Anomalous Hall Effect (AHE) in 2D metals

\[ \sigma^H = \frac{e^2}{2\pi \hbar} \sum_n \nu_n \]

\[ \nu_n = \frac{1}{2\pi} \int \mathcal{A}_{n}^{a}(k_F) dk_{Fa} = \frac{\phi_F}{2\pi}. \text{ modulo an integer} \]

\[ e^{i\phi_F} \]

Berry phase factor for moving a quasiparticle around the Fermi surface

In principle, derived to all orders in diagrammatic perturbation theory
• The following arguments suggests that the same result can be obtained for composite fermions, without any perturbative path from free electrons.

• This is evidence in favor of the conjecture that the result is quite general, and fully non-perturbative.

• However, it may miss some details involving the topological degeneracy, as it disagrees with arguments of Son and others, and may need some changes.
The real-space orbit of the dipolar electron displacement around the flux-attachment center of a fixed correlation hole (held in place by the other electrons) exactly tracks the shape of the Fermi surface:

\[
(k\text{-space area of Fermi surface}) \times \ell_B^2 = (\text{area of real-space orbit}) / \ell_B^2
\]
given by Luttinger theorem.

Berry phase:

\[
e^{i\phi} = e^{2\pi i v}
\]

Berry phase = Bohm-Aharonov phase for e to move around real space path around origin fixed at the center of the correlation hole.

in agreement with

\[
\sigma_H = \frac{e^2}{\hbar} \frac{\phi}{2\pi}
\]

FDMH 2005

composite fermion

Fermi surface
• for half-filled Landau level, this gives a Berry phase of $\pi$

• But this comes from a universal uniform Berry curvature (and not from a Dirac cone with a “$\mathbb{Z}_2$” singularity at “$k = 0$” (Son)).

• “$k = 0$” is the state of the cf with inversion symmetry and no electric dipole moment. Its energy is quadratic in the dipole moment (and thus in the momentum).

**microscopic picture**

- quadratic dispersion
- uniform k-space Berry curvature inside and outside the Fermi surface

**Son’s “Dirac” picture**

- linear dispersion
- $\mathbb{Z}_2$ point singularity at “$k = 0$”
- No k-space Berry curvature
Model for $1/m$ CFL states

- choose distinct “occupied orbitals” (allowed dipole moments, quantized by the pbc)

$$\{d_i, i = 1, \ldots N\} \in \{\frac{L}{N}\}$$

which minimize

$$\frac{1}{N} \sum_{i<j} |d_i - d_j|^2 = \frac{1}{2} \sum_i |d_i - \bar{d}|^2$$

for fixed

$$\bar{d} = \frac{1}{N} \sum_i d_i$$

- $\bar{d} \mod \{\frac{L}{N}\}$ is a many-body quantum number that takes $N^2$ distinct values. There is thus one such configuration per sector of this many-body translational quantum number.
The model 1/m CFL states (including the boson case $m = 1$) are

$$
\Psi(\{z_i, z_i^*\}, \{d_i\}, \{w_\alpha\}) \propto \left( \det M_{i,j}(\{z_k\}, d_j, d_j^*, \bar{d}) \right) \times \left( \prod_{i<j} \sigma(z_i - z_j) \right)^{m-2} \prod_{\alpha=1}^m \sigma((\sum_i z_i) - w_\alpha) \prod_{i=1}^N e^{-\frac{1}{4} \frac{z_i^* z_i}{\ell_B^2}}
$$


The matrix in the determinant is

$$M_{i,j}(\{z_k\}, d_j, d_j^*, \bar{d}) = e^{\frac{1}{m} \frac{d_j^* z_i}{2\ell_B^2}} \prod_{k \neq i} \sigma(z_i - z_k - d_j + \bar{d})$$

also:

$$\sum_{\alpha=1}^m w_\alpha = \sum_{j=1}^N d_j = N\bar{d}$$

mean value of $d_j$

(a continuously adjustable parameter)

complex cf dipoles $e d_j$

($d_j$ is quantized in units $\frac{L}{N}$)
• Now we see that the “Fermi sea” is invariant under uniform translation in “dipole space”

# Z_{COM} overlap with PH-conjugate in opposite charge sector 1-
overlap
0 0.999998870263 1.1297367517e-06
1 0.999999369175 6.3082507884e-07
2 0.99999860296 1.39704033186e-06
3 0.99999860296 1.3970403312e-06
4 0.999999369175 6.30825078063e-07
5 0.999998870263 1.12973675237e-06
6 0.999999369175 6.30825079173e-07
7 0.99999860296 1.39704032942e-06
8 0.99999860296 1.39704032909e-06
9 0.999999369175 6.30825078507e-07

Computing ph symmetry
(with Scott Geraedts)
model state is numerically very close to p-h symmetry

cluster of adjacent occupied states
Q: is there a clear test to distinguish the two pictures?

A: **Yes**! Compute the Berry phase for evolution of a cf quasiparticle around a **contractible** loop that stays close to the Fermi surface, but encloses a finite k-space area.

- **Prediction of argument given here**: Berry phase proportional to k-space area enclosed by path.

- Son’s prediction: no Berry phase \((\text{at } \nu = \frac{1}{2})\)
Q: Can one not just define “Dirac” to mean “having a Berry phase of $\pi$ for going around the closed (2D) Fermi surface arc?”

A: No! “Dirac” has a specific meaning: a conical band-touching point which in 2D only occurs by “$\mathbb{Z}_2$” topological protection (requires states in the orthogonal ensemble with inversion and time-reversal symmetry) and has no Berry curvature. Without this feature, the system cannot be called Dirac*.

*to argue that anyone is free to redefine “Dirac” as they wish is an invitation to indulge in a kind of “product-labeling violation”: in other areas (watches, handbags,..) this can have severe legal consequences!
This is the **entire** problem: nothing other than this matters!

- H has translation and inversion symmetry

\[
[(R_1^x + R_2^x), (R_1^y - R_2^y)] = 0
\]

\[
[H, \sum_i R_i] = 0
\]

- generator of translations and electric dipole moment!

\[
[(R_1^x - R_2^x), (R_1^y - R_2^y)] = -2i\ell_B^2
\]

- relative coordinate of a pair of particles behaves like a single particle

\[
H = \sum_{i<j} U(R_i - R_j)
\]

\[
[R^x, R^y] = -i\ell_B^2
\]

like phase-space, has Heisenberg uncertainty principle

want to avoid this state

two-particle energy levels

gap
• Solvable model! ("short-range pseudopotential")

\[
U(r_{12}) = \left( A + B \left( \frac{(r_{12})^2}{\ell_B^2} \right) \right) e^{-\frac{(r_{12})^2}{2\ell_B^2}}
\]

• Laughlin state

\[
|\Psi^m_L\rangle = \prod_{i<j} (a_i^{\dagger} - a_j^{\dagger})^m |0\rangle
\]

\[
a_i |0\rangle = 0 \quad a_i^{\dagger} = \frac{R^x + iR^y}{\sqrt{2\ell_B}}
\]

\[
E_L = 0 \quad [a_i, a_j^{\dagger}] = \delta_{ij}
\]

maximum density null state

• \(m=2\): (bosons): all pairs avoid the symmetric state \(E_2 = \frac{1}{2}(A+B)\)

• \(m=3\): (fermions): all pairs avoid the antisymmetric state \(E_2 = \frac{1}{2}B\)