Invariants in the Presence of Disorder

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Work in collaboration with Hermann Schulz-Baldes

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CAREER-DMR-1056168
The results apply to the phases classified by $\mathbb{Z}$

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The topology is encoded in

- $P_F = \chi(H\omega \leq E_F)$ (Fermi Projection) for even dimensions
- $U_F$ (Fermi unitary operator) for odd dimensions

$$P_F = \frac{1}{2} \begin{pmatrix} 1 & -U_F^* \\ -U_F & 1 \end{pmatrix}$$
Few Words About the Chern Numbers

On the $\mathbb{T}^d$ torus:

\[ P_k^2 = P_k^* = P_k \]
\[ U_k U_k^* = U_k^* U_k = I \]

\( l = \{ i_1, \ldots, i_n \} \subset \{ 1, 2, \ldots, d \} \) an ordered set of indices (\(|l| = n \leq d\))

\[
\text{Ch}_l(P) = \Lambda_l \sum_{\rho \in S_d} (-1)^\rho \int_{\mathbb{T}^d} dk \ P_k \prod_{i=1}^{\lvert l \rvert} \frac{\partial P_k}{\partial k_{\rho_i}} \quad (\lvert l \rvert = \text{even})
\]

\[
\text{Ch}_l(U) = \Lambda_l \sum_{\rho \in S_d} (-1)^\rho \int_{\mathbb{T}^d} dk \ \prod_{i=1}^{\lvert l \rvert} U_k^* \frac{\partial U_k}{\partial k_{\rho_i}} \quad (\lvert l \rvert = \text{odd})
\]
Real Space Representation

$X$ is the position operator on the $\mathbb{Z}^d$-lattice

$$Ch_I(P) = \Lambda_I \sum_{\rho \in S_d} (-1)^\rho \mathcal{T} \left( P \prod_{i=1}^d i [P, X_{\rho_i}] \right) \quad (|I| = \text{even})$$

$$Ch_I(U) = \Lambda_I \sum_{\rho \in S_d} (-1)^\rho \mathcal{T} \left( \prod_{i=1}^d i U^* [U, X_{\rho_i}] \right) \quad (|I| = \text{odd})$$

Make sense in the presence of magnetic fields and disorder ... but why are they invariant and where do they take values?

Is there a bulk-boundary correspondence for all of them?

What are the physical implications?
Homogeneous Lattice Hamiltonians (with B-field)

The most general form:

\[ H : \mathbb{C}^N \otimes \ell^2(\mathbb{Z}^d) \rightarrow \mathbb{C}^N \otimes \ell^2(\mathbb{Z}^d) \]

\[ H_\omega(B) = \sum_{y \in \mathcal{R}} \sum_{x \in \mathbb{Z}^d} W_y(\tau_x \omega) \otimes |x\rangle \langle x| U_y \]

\( \mathcal{R} \subset \mathbb{Z}^d = \) finite hopping range; \( U_y = e^{i\gamma^x S_y} = \) magnetic translations

Covariant property:

\[ U_y H_\omega U_y^* = H_{\tau_y \omega} \]

w.r.t. a discrete ergodic dynamical system (\( \Omega, \tau, \mathbb{Z}^d, \mathbb{P} \)).
Homogeneous Lattice Hamiltonians with an Edge

\[ \hat{H}_\omega : \mathbb{C}^N \otimes \ell^2(\mathbb{Z}^{d-1} \times \mathbb{N}) \to \mathbb{C}^N \otimes \ell^2(\mathbb{Z}^{d-1} \times \mathbb{N}) \]

\[ \hat{H}_\omega (B) = \Pi_d H_\omega (B) \Pi_d^* + \tilde{H}_\omega (B) \]

\[ \tilde{H}_\omega (B) = \sum_{n,m=0}^{R} \sum_{y \in \mathcal{R}'} \sum_{x \in \mathbb{Z}^{d-1}} \tilde{W}_{nm}^y (\tau_x, n\omega) \otimes |x, n\rangle \langle x, n| U_{y,n-m} \]
The Algebra of Bulk Physical Observables $\mathcal{A}_d$

Definition

The universal $C^*$-algebra (i.e. equipped with a norm $\|aa^*\| = \|a\|^2$)

$$\mathcal{A}_d = C^*\left( C(\Omega), u_1, \ldots, u_d \right),$$

generated by the following commutation relations:

$$u_i u_i^* = u_i^* u_i = 1, \quad i = 1, \ldots, d$$

$$u_i u_j = e^{i\theta_{ij}} u_j u_i, \quad i, j = 1, \ldots, d \quad \text{non-commutative torus}$$

$$\phi u_j = u_j (\phi \circ \tau_j), \quad \forall \phi \in C(\Omega), \ j = 1, \ldots, d.$$

A generic element takes the form

$$a = \sum_{x \in \mathbb{Z}^d} a(\omega, x) u_x, \quad u_x = u_1^{x_1} \cdots u_d^{x_d}, \quad x = (x_1, \ldots, x_d).$$
Algebra of Half-Space Observables

Definition
The universal C*-algebra

\[ \hat{A}_d = C^*(C(\Omega), \hat{u}_1, \ldots, \hat{u}_d) \]

generated by the commutation relations

\[
\begin{align*}
\hat{u}_d^* \hat{u}_d &= 1, & \hat{u}_d \hat{u}_d^* &= 1 - \hat{e}, & \text{(only difference!)} \\
\hat{u}_i \hat{u}_i^* &= \hat{u}_i^* \hat{u}_i = 1, & i &= 1, \ldots, d - 1, \\
\hat{u}_i \hat{u}_j &= e^{i \theta_{i,j}} \hat{u}_j \hat{u}_i, & i, j &= 1, \ldots, d, \\
\phi \hat{e} &= \hat{e} \phi, & \phi \hat{u}_j &= \hat{u}_j(\phi \circ \tau_j), & \forall \phi &\in C(\Omega), & j &= 1, \ldots, d
\end{align*}
\]

A generic element takes the form

\[
\hat{a} = \sum_{m,n \in \mathbb{N}} \sum_{x \in \mathbb{Z}^{d-1}} \hat{a}_{nm}(\omega, x) \hat{u}_x \hat{u}_d^n(\hat{u}_d^*)^m.
\]
Definition

The algebra of boundary physical observables is defined as the double sided ideal generated by $\hat{e}$

$$\mathcal{E}_d = \hat{A}_d \hat{e} \hat{A}_d$$

A generic element takes the form

$$\tilde{a} = \sum_{m,n \in \mathbb{N}} \sum_{x \in \mathbb{Z}^{d-1}} \tilde{a}_{nm}(\omega, x) \hat{u}_x \hat{u}_d^n \hat{e}(\hat{u}_d^*)^m.$$  

Proposition

We have the exact sequence, which is split at the level of linear spaces:

$$0 \longrightarrow \mathcal{E}_d \overset{i}{\longrightarrow} \hat{A}_d \overset{ev}{\longrightarrow} A_d \overset{i'}{\longrightarrow} 0$$
A Few Words about K-Groups

\( K_0(\mathcal{A}) \) classifies the projections \((p^2 = p^* = p)\) from \( \mathbb{K} \otimes \mathcal{A} \) under the homotopy equivalence:

\[
\begin{pmatrix}
p & 0 \\
0 & 0 
\end{pmatrix} \sim_h \begin{pmatrix}
p' & 0 \\
0 & 0 
\end{pmatrix}, \quad [p]_0 + [p']_0 = \begin{pmatrix}
p & 0 \\
0 & p' 
\end{pmatrix}_0
\]

\( K_1(\mathcal{A}) \) classifies the unitaries \((uu^* = u^*u = 1)\) from \( \mathbb{K} \otimes \mathcal{A} \) under the homotopy equivalence:

\[
\begin{pmatrix}
u & 0 \\
0 & 1 
\end{pmatrix} \sim_h \begin{pmatrix}
u' & 0 \\
0 & 1 
\end{pmatrix}, \quad [u]_1 + [u']_1 = [uu']_1
\]

Important: Separable \( C^* \)-algebras have countable \( K \)-groups, hence they lead to sensible classifications.
The Six-Term Exact Sequence

The short exact sequence

\[ 0 \rightarrow \mathcal{E}_d \xrightarrow{i} \hat{A}_d \xrightarrow{\text{ev}} A_d \rightarrow 0 \]

induces a six-term exact sequence

\[ K_0(\mathcal{E}_d) \xrightarrow{i_\ast} K_0(\hat{A}_d) \xrightarrow{\text{ev}_\ast} K_0(A_d) \]

\[ \text{Ind} \]

\[ K_1(\mathcal{E}_d) \xrightarrow{\text{ev}_\ast} K_1(\hat{A}_d) \xrightarrow{i_\ast} K_1(A_d) \]

\[ \text{Exp} \]
The K-Groups and their Generators (Ω assumed contractible)

For $d \geq 2$, the $K$-groups of the observable algebras are given by

$$K_j(\mathcal{A}_d) = K_j(\mathcal{E}_{d+1}) = K_j(\widehat{\mathcal{A}}_{d+1}) = \mathbb{Z}^{2^d-1}, \quad j = 0, 1,$$

and all the generators are known explicitly.

For $K_0(\mathcal{A})$, the generators are labeled like $\{[p_J]\}_{J \subset \{1, \ldots, d\}}$, $|J| = \text{even}$:

$$p \sim_h \bigoplus_{|J| \text{ even}} \alpha_J p_J, \quad \alpha_J \in \mathbb{N}$$

For $K_1(\mathcal{A})$, the generators are labeled like $\{[u_J]\}_{J \subset \{1, \ldots, d\}}$ and $|J| = \text{odd}$:

$$u \sim_h \bigoplus_{|J| \text{ odd}} \beta_J u_J, \quad \beta_J \in \mathbb{N}$$
Encoding the Topology in the Boundary (Even Case)

\[
\begin{align*}
K_0(E_d) & \xrightarrow{i_*} K_0(\hat{A}_d) & \xrightarrow{\text{ev}_*} & K_0(A_d) \\
\text{Ind} & & \text{Exp} \\
K_1(A_d) & \xleftarrow{\text{ev}_*} K_1(\hat{A}_d) & \xleftarrow{i_*} & K_1(E_d)
\end{align*}
\]

Bulk: \( h \in M_N(\mathbb{C}) \otimes A_d, \ h = h^*, \ \epsilon_F \) located anywhere in the bulk gap.

\[
p_F = \chi(h \leq \epsilon_F) \in A_d \rightarrow [p_F]_0 \in K_0(A_d)
\]

Boundary: \( \hat{h} = i'(h) + \tilde{h} \in M_N(\mathbb{C}) \otimes \hat{A}_d \)

\[
\text{Exp}[p_F]_0 = [\exp (2\pi \mathbb{i} G(\hat{h}))]_1
\]

\[
\tilde{u}_\Delta = \exp (2\pi \mathbb{i} G(\hat{h})) \in E_d \rightarrow [\tilde{u}_\Delta]_1 \in K_1(E_d)
\]

\( G \) smooth and \( G = 1, 0 \) below/above \([\epsilon_F - \delta, \epsilon_F + \delta]\).
Encodings the Topology in the Boundary (Odd Case)

Bulk: $h \in M_{2N}(\mathbb{C}) \otimes \mathcal{A}_d$, $h = h^*$, \( \begin{pmatrix} 1_N & 0 \\ 0 & -1_N \end{pmatrix} \) $h \begin{pmatrix} 1_N & 0 \\ 0 & -1_N \end{pmatrix} = -h$, $\epsilon_F = 0$

\[
\text{sgn}(h) = \begin{pmatrix} 0 & u_F^* \\ u_F & 0 \end{pmatrix} \rightarrow [u_F]_1 \in K_1(\mathcal{A}_d)
\]

Boundary: $\hat{h} = i'(h) + \tilde{h} \in M_N(\mathbb{C}) \otimes \hat{\mathcal{A}}_d$

\[
\text{Ind}[u_F]_1 = [\tilde{p}_\Delta]_0
\]

\[
\tilde{p}_\Delta = e^{-i \frac{\pi}{2} G(h)} \text{diag}(1_N, 0_N) e^{i \frac{\pi}{2} G(h)} \in \mathcal{E}_d \rightarrow [\tilde{p}_\Delta]_0 \in K_0(\mathcal{E}_d)
\]

$G$ smooth, odd and $G = \pm 1$ above/below $[-\delta, \delta]$. 
Cyclic Cocycles

Cyclic \((n + 1)\)-linear functionals over \(A\) (whose space is denoted by \(C^n(A)\))

\[
\varphi(a_1, \ldots, a_n, a_0) = (-1)^n \varphi(a_0, a_1, \ldots, a_n) \in \mathbb{C}
\]

Hochschild co-boundary map \((b \circ b = 0)\)

\[
b \varphi(a_0, a_1, \ldots, a_{n+1}) = \sum_{j=0}^{n} (-1)^j \varphi(a_0, \ldots, a_j a_j+1, \ldots a_{n+1})
\]

\[
+ (-1)^{n+1} \varphi(a_{n+1} a_0, \ldots, a_n),
\]

Cyclic cohomology is defined by the cohomology of the complex

\[
\ldots \xrightarrow{b} C^{n-1}(A) \xrightarrow{b} C^n(A) \xrightarrow{b} \ldots,
\]
Cyclic cocycles
Defined as the elements of the space \( \text{Ker} \ b/\text{Im} \ b \) (or \( b \phi = 0 \))

Even cocycles \((n = \text{even})\) pair with \( K_0 \)-group
\[
\langle [\phi], [p]_0 \rangle = (\text{Tr} \# \phi)(p, \ldots, p) \in \mathbb{C}
\]

Odd cocycles \((n = \text{odd})\) pair with \( K_1 \)-group
\[
\langle [\phi], [u]_1 \rangle = (\text{Tr} \# \phi)(u^* - 1, u - 1, \ldots, u - 1) \in \mathbb{C}
\]

Invariance of the righthand sides w.r.t. both \( \phi \) and \( p \) or \( u \) is essential for the bulk-boundary correspondence.
Non-Commutative Calculus

**Bulk Algebra:** For a generic element  \( \sum_{x \in \mathbb{Z}^d} a(\omega, x) u^x \)

\[
\partial_j a = \iota \sum_{x \in \mathbb{Z}^d} x_j a(\omega, x) u^x, \quad j = 1, d, \quad T(a) = \int_\Omega \mathbb{P}(d\omega) \, a(\omega, 0).
\]

**Edge Algebra:** For a generic element  \( \sum_{n,m \in \mathbb{N}} \sum_{x \in \mathbb{Z}^{d-1}} \tilde{a}_{nm}(\omega, x) \hat{u}^x \hat{u}_d^n \hat{e}(\hat{u}_d^*)^m \)

\[
\tilde{\partial}_j \tilde{a} = \iota \sum_{n,m \in \mathbb{N}} \sum_{x \in \mathbb{Z}^{d-1}} x_j \tilde{a}_{nm}(\omega, x) \hat{u}^x \hat{u}_d^n \hat{e}(\hat{u}_d^*)^m, \quad j = 1, d - 1,
\]

\[
\tilde{T}(\tilde{a}) = \sum_{n \in \mathbb{N}} \int_\Omega \mathbb{P}(d\omega) \, \tilde{a}_{nn}(\omega, 0).
\]
Bulk Numerical Invariants

Proposition (R. Nest, 1988)

The cyclic cocycles (called here the Chern cocycles)

$$\xi_I(a_0, \ldots, a_n) = \Lambda_n \sum_{\rho \in S_n} (-1)^\rho \mathcal{T}\left(a_0 \prod_{j=1}^n \partial_{\rho_j} a_j\right), \quad I \subset \{1, \ldots, d\}, \quad n = |I|$$

generate the entire (periodic) cyclic cohomology of $A_d$.

Even Chern numbers ($|I|$ = even)

$$\text{Ch}_I(p_F) = \langle [\xi_I], [p_F]_0 \rangle$$

Odd Chern numbers ($|I|$ = odd)

$$\text{Ch}_I(u_F) = \langle [\xi_I], [u_F]_1 \rangle$$
Boundary Numerical Invariants

Proposition (R. Nest, 1988)

The cyclic cocycles

$$\widetilde{\xi}_I(\tilde{a}_0, \ldots, \tilde{a}_n) = \Lambda_n \sum_{\rho \in S_n} (-1)^\rho \widetilde{T} \left( \tilde{a}_0 \prod_{j=1}^n \tilde{\partial}_\rho \tilde{a}_j \right), \quad I \subset \{1, \ldots, d-1\}, \quad n = |I|$$

generate the entire (periodic) cyclic cohomology of $E_d$.

Boundary Odd Chern numbers ($|I| = \text{odd}$)

$$\widehat{\text{Ch}}_I(\tilde{\Delta}) = \langle [\tilde{\xi}_I], [\tilde{\Delta}]_1 \rangle$$

Boundary Even Chern numbers ($|I| = \text{even}$)

$$\widehat{\text{Ch}}_I(\tilde{\rho}_\Delta) = \langle [\tilde{\xi}_I], [\tilde{\rho}_\Delta]_0 \rangle$$
Duality of Pairings

Theorem (Kellendonk et al, 2002)

\[ K_0(\mathcal{E}_d) \xrightarrow{i_*} K_0(\hat{\mathcal{A}}_d) \xrightarrow{\text{ev}_*} K_0(A_d) \]
\[ K_1(A_d) \xleftarrow{\text{Ind}} K_1(\hat{\mathcal{A}}_d) \xleftarrow{i_*} K_1(\mathcal{E}_d) \]

For any set of indices \( I \) such that \( d \notin I \) and \(|I| \) odd,

\[ \langle [\xi_{I\cup\{d\}}], [p]_0 \rangle = -\langle \tilde{\xi}_I, \text{Exp}[p]_0 \rangle, \]

while for \(|I| \) even,

\[ \langle [\xi_{I\cup\{d\}}], [u]_1 \rangle = \langle [\tilde{\xi}_I], \text{Ind}[u]_1 \rangle. \]
Equality Between Bulk and Boundary Topological Invariants

Theorem

(i) Let \( \hat{h} = i'(h) + \tilde{h} \in M_N(\mathbb{C}) \otimes \hat{A}_d \) satisfying bulk gap hypothesis, and let \( I \) be a set of indices such that \( |I| = 2k - 1 < d \) and \( d \notin I \). Then:

\[
\text{Ch}_{I \cup \{d\}}(p_F) = \tilde{\text{Ch}}_I(\tilde{u}_\Delta).
\]

(ii) Let \( \hat{h} = i'(h) + \tilde{h} \in M_{2N}(\mathbb{C}) \otimes \hat{A}_d \) satisfying the bulk gap hypothesis and the chiral symmetry. Let \( I \) be a set of indices such that \( |I| = 2k < d \) and \( d \notin I \). Then:

\[
\text{Ch}_{I \cup \{d\}}(u_F) = \tilde{\text{Ch}}_I(\tilde{p}_\Delta).
\]
## Bulk:

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<th>75% Disorder</th>
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**Table:** Bulk Chern number for disordered Haldane model. The amplitude of the disorder is given as percentage of the bulk clean gap $\Delta_0$. The spectral gap closes for 100% disorder, but here we consider only regimes with positive spectral gap. The data was averaged over 10 independent runs with randomly updated disorder configurations.

## Boundary:

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<tr>
<td>151</td>
<td>0.999999999976±1.97E-14</td>
<td>1.000000000641±1.64E-11</td>
<td>0.999999999998685±6.52E-12</td>
</tr>
</tbody>
</table>

**Table:** Boundary Chern number with open boundaries in the 2-nd direction, evaluated numerically on ribbon-shaped lattices of size $Q_1 \times 51$. The data represent the average plus/minus the standard deviation as determined from 20 independent runs with randomly updated disorder configurations.
Generalized Streda Formulas

Classic Streda Formula:

$$\sigma_H = \partial_B n_e \text{ or } \langle \xi_{1,2}, p_F \rangle = \partial_{\theta_{1,2}} \langle \xi_0, p_F \rangle.$$

Theorem

(i) If $|I|$ is even and $e$ is a projection smooth w.r.t. $B$-field,

$$\partial_{\theta_{i,j}} \langle [\xi_I], [p]_0 \rangle = \frac{1}{2\pi} \langle [\xi_{\{i,j\}\cup I}], [p]_0 \rangle.$$

(ii) If $|I|$ is odd and $u$ is a unitary smooth w.r.t. $B$-field,

$$\partial_{\theta_{i,j}} \langle [\xi_I], [u]_1 \rangle = \frac{1}{2\pi} \langle [\xi_{\{i,j\}\cup I}], [u]_1 \rangle.$$
Let \( I, J \subset \{1, \ldots, d\} \) be increasingly ordered sets of even cardinality. Then the generators \( p_J \) of \( K_0(\mathcal{A}_d) \) and the cocycles \( \xi_I \) pair as follows:

\[
\langle [\xi_I], [p_J]_0 \rangle = \begin{cases} 
0, & I \setminus J \neq \emptyset, \\
1, & I = J, \\
(2\pi)^{-\frac{1}{2}|J\setminus I|} \text{Pf}(\theta_{J\setminus I}), & I \subset J.
\end{cases}
\]

Similarly, for index sets \( I \) and \( J \) of odd cardinality and generators \( u_J \) of \( K_1(\mathcal{A}_d) \)

\[
\langle [\xi_I], [u_J]_1 \rangle = \begin{cases} 
0, & I \setminus J \neq \emptyset, \\
1, & I = J, \\
(2\pi)^{-\frac{1}{2}|J\setminus I|} \text{Pf}(\theta_{J\setminus I}), & I \subset J.
\end{cases}
\]
The range of the pairings and higher gap labeling

Corollary

The image \( \text{Range}(\xi_I) \) of the index pairings with a cocycle \( \xi_I \) on \( \mathcal{A}_d \) is given by

\[
\text{Range}(\text{Ch}_I) = \mathbb{Z} + \sum_{I \subset J} (2\pi)^{-\frac{1}{2}|J \setminus I|} \sum_{PP(J \setminus I)} \theta_{p_1} \cdots \theta_{p_{\frac{1}{2}|J \setminus I|}} \mathbb{Z},
\]

where the sum goes only over \( J \)'s with \( |J \setminus I| \) even.
Physical Interpretation of the Bulk Invariants

All Chern numbers (odd or even) can be obtained by taking derivatives w.r.t. the fluxes $\theta$ on one of the following cocycles

\begin{enumerate}
\item $\langle \xi\{i,j\}, p_F \rangle = \sigma_{ij} \quad \text{(Conductivity tensor)}$
\item $\langle \xi\{i\}, u_F \rangle = P_i^C \quad \text{(Vector of Chiral Polarization)}$
\end{enumerate}

Even Chern numbers give the linear and non-linear magneto-electric transport coefficients.

Odd Chern numbers give the linear and non-linear magneto-electric response coefficients.
Physical Interpretation of the Boundary Invariants

All Chern numbers (odd or even) can be obtained by taking derivatives w.r.t. the fluxes \( \theta \) on one of the following cocycles

1. \( \langle \tilde{\xi}_j, \tilde{u}_\Delta \rangle = \tilde{G}_j \) (linear conductance of the boundary)

2. \( \langle \tilde{\xi}_{i,j}, \tilde{p}_\Delta \rangle = \tilde{\sigma}^C_{ij} \) (chiral Hall conductivity tensor of the boundary)

Odd Chern numbers give the linear and non-linear conductance of the boundary.

Even Chern numbers give the linear and non-linear chiral conductivity tensors.
A piezo-magneto-electric device

Consider the following representation of $\mathcal{A}_4$

$$\pi(u_j) = U_j, \quad \text{for } j = 1, 2, 3, \quad \pi(u_4) = e^{i\vec{\theta}_4 \cdot \vec{X}},$$

where $\vec{\theta}_4 = (\theta_{1,4}, \theta_{2,4}, \theta_{3,4})$. Take

$$h = \sum_{j=1}^{4} (u_j + u_j^*), \quad \text{Ch}_4(p_F) \neq 0.$$

Let $H = \pi(h)$ on $\ell^2(\mathbb{Z}^3)$:

$$H = \sum_{j=1}^{3} (U_j + U_j^*) + 2 \cos \langle \theta_4, X \rangle$$

which describes a crystal in a magnetic field and an additional incommensurate periodic potential. Using the generalized Streda formula

$$\text{Ch}_4(p_F) = 2\pi \partial_{\theta_{3,4}} \sigma_{12}$$

Hence, if we periodically squeeze the crystal in the 3rd direction it will display the Hall effect in the (1,2) plane.
References


These newer works build on the following classic results:


Canonical Representation on $\ell^2(\mathbb{Z}^d)$

Proposition

Let $U_j$ be the elementary magnetic shifts in the Landau gauge. Then

$$\pi_\omega(u_j) = U_j, \quad j = 1, \ldots, d,$$

and

$$\pi_\omega(\phi) = \sum_{x \in \mathbb{Z}^d} \phi(\tau_x \omega) |x\rangle\langle x|, \quad \forall \, \phi \in C(\Omega),$$

defines a family $\{\pi_\omega\}_{\omega \in \Omega}$ of faithful representations.

For generic elements

$$\mathcal{A}_d \ni a = \sum_{y \in \mathbb{Z}^d} a(\omega, y) u_y \longrightarrow \pi_\omega(a) = \sum_{x, y \in \mathbb{Z}^d} a(\tau_x \omega, y) |x\rangle\langle x| U_y.$$ 

All homogeneous lattice models can be generated from $M_N(\mathbb{C}) \otimes \mathcal{A}_d$!
Canonical Representation on $\ell^2(\mathbb{Z}^{d-1} \times \mathbb{N})$

Proposition

$$\hat{\pi}_\omega(\hat{u}_j) = \Pi_d \pi_\omega(u_j) \Pi_d^*, \quad j = 1, \ldots, d$$

$$\hat{\pi}_\omega(\phi) = \Pi_d \pi_\omega(\phi) \Pi_d^*, \quad \forall \phi \in C(\Omega)$$

$$\hat{\pi}_\omega(\hat{e}) = P_{\hat{e}} = \sum_{y \in \mathbb{Z}^{d-1}} |y, 0\rangle \langle y, 0|$$

defines a family $\{\hat{\pi}_\omega\}_{\omega \in \Omega}$ of faithful representations.

For a generic element from $\hat{A}_d$

$$\hat{a} = i'(a) + \tilde{a} \quad \text{(split exactness)}$$

$$\hat{\pi}_\omega(\hat{a}) = \Pi_d \pi_\omega(a) \Pi_d^* + \sum_{n,m \in \mathbb{N}} \sum_{x,y \in \mathbb{Z}^{d-1}} \tilde{a}_{nm}(\tau_x, n\omega, y)|x, n\rangle \langle x, n| U_{y, n-m}$$

Generates all homogeneous lattice models with an edge from $M_N(\mathbb{C}) \otimes \hat{A}_d$!
Anomalous Hall effect at the surface of chiral insulators

The boundary projector

\[ \tilde{p}_\Delta = e^{-i \frac{\pi}{2} \hat{G}(\hat{h})} \text{diag}(1_N, 0_N) e^{i \frac{\pi}{2} \hat{G}(\hat{h})} = J_+ p_\delta J_+ - J_- p_\delta J_- \]

The surface invariant

\[ \tilde{\text{Ch}}_2(\tilde{p}_\Delta) = \tilde{\text{Ch}}_2(p_\delta^+) - \tilde{\text{Ch}}_2(p_\delta^-) \]

Chern number of the spectral projector

\[ \tilde{\text{Ch}}_2(p_\delta) = \tilde{\text{Ch}}_2(p_\delta^+) + \tilde{\text{Ch}}_2(p_\delta^-) \neq 0 \quad \text{provided} \quad \tilde{\text{Ch}}_2(\tilde{p}_\Delta) = \text{odd} \]
• IQHE is more about horizontal than the vertical

• The horizontal label is electron density and not $E_F$!

• Or the magnetic field and electron density is kept constant!

• The Fermi energy must reside in the essential spectrum.

\[ n_e = \mathcal{T}(P_F) \]
Why Strong Disorder is Needed?

\[ \frac{\Phi_{12}}{\Phi_0} \]

\[ \sigma_{12} = 0 \]

\[ \sigma_{12} = 1 \]
Hall Plateaus Explained (Bellissard et al (1994))

Let $H_\omega$ be a disordered Hamiltonian (say Hofstadter with on-site disorder). Let $P_\omega$ be the projector onto the states below $E_F$ (the Fermi projection). If

$$
\sum_{x \in \mathbb{Z}^2} \int_{\Omega} d\mathbb{P}(\omega) |\langle 0 | P_\omega | x \rangle|^2 < \infty \quad \text{(optimal condition!)}
$$

then

$$
P_\omega \frac{X_1 + iX_2}{\sqrt{X_1^2 + X_2^2}} P_\omega
$$

is a Fredholm operator $\mathbb{P}$-almost surely and

$$
\sigma_{\text{Hall}} = 2\pi i \mathcal{T} \left( P_\omega \left[ [X_1, P_\omega], [X_2, P_\omega] \right] \right) = \text{Index} \left( P_\omega \frac{X_1 + iX_2}{\sqrt{X_1^2 + X_2^2}} P_\omega \right).
$$