Weak solutions to the Einstein equations in spherical or T2 symmetry

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1. Objective: assume symmetry, self-gravitating compressible fluids, shock waves, metric with weak regularity

2. Future vanishing area property for compressible fluids in Gowdy symmetry (PLF-Rendall 2010, Grubic-PLF 2014)


4. Formation of trapped surfaces for compressible fluids in spherical symmetry (Burtscher-PLF 2014)
1. Objective

- spherical symmetry (SO(3) isometry group action)
- T2 symmetry (T2 isometry group action)
- Gowdy/plane symmetry (vanishing twists/polarization)

Maximal hyperbolic developments and future boundary

- 1+1 nonlinear wave systems, rich global dynamics, large data
- Foliations, sextension principles, late-time asymptotics
- Geodesic completeness, censorship conjectures (generic data)

Motivations

- Christodoulou (around 1990), assuming spherical symmetry
  - Scalar field or two-phase stiff fluid, BV (bounded variation) regularity
  - Settled positively the weak cosmic censorship
- Dafermos (2003, completeness of future null infinity), ...
  - Kommemii (2012, tame matter)
- T2 symmetry: \( C^2 \) regularity
  - vacuum: Berger, Chruściel, Isenberg, Moncrief, Ringström
  - kinetic matter: Rendall, Rein, Dafermos, Smulevici, Andréasson
Achievements

- Include compressible matter
- Encompass metrics with weak regularity

Challenges

- revisit techniques and results for \((1 + 1)\) problems
- Weak solutions
- Compressible matter
- Vacuum spacetimes

Vacuum / matter

- Vacuum spacetimes
  - curvature singularities on null hypersurfaces
  - colliding spacetimes à la Khan-Penrose
- Compressible matter
  - discontinuity\(^1\) hypersurfaces (propagation at about the sound speed)
  - implying curvature discontinuities, shock interactions
  - scalar field: the special case of irrotational null fluids

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\(^1\)P.D. Lax, F. John, D. Christodoulou, J. Speck
Formulation in the distributional sense

- **Einstein equations in the weak sense**\(^2\)
  - Coordinate-independent notion
  - Metric \( g \in W^{1,2} \cap L^\infty \) with \( g^{-1} \in L^\infty \)
  - Schematically, for the curvature \( Rc = \partial \Gamma + \Gamma \star \Gamma \) where \( \Gamma \in L^2 \)
  - Foliations with second fundamental form in \( L^2 \)

- **Symmetry condition in the weak sense**
  - Lie derivative \( \mathcal{L}_Z h \) of some 2-tensor \( h \) in the sense of distributions:
    for any \( C^1 \) vector fields \( X, T, Z \)
    \[
    (\mathcal{L}_X h)(T, Z) := X(h(T, Z)) - h(\mathcal{L}_X T, Z) - h(T, \mathcal{L}_X Z)
    \]

**Without symmetry assumption**

- Piecewise regular spacetimes with localized singularities
- Nonlinear interactions of impulsive gravitational waves for the vacuum Einstein equations
- Related: “essential weak null” singularities not \( W^{1,2} \) (Dafermos 2015)

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\(^2\)ArXiv gr-qc: 0712.0122 P. LeFloch and C. Mardare, Definition of spacetimes with distributional curvature
2. Future vanishing area property

Compressible fluids in Gowdy symmetry\(^3\)

- **Einstein equations**
  \[
  R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = T_{\alpha\beta}
  \]

- **Compressible fluids**
  \[
  T_{\alpha\beta} = (\mu + p(\mu)) u_{\alpha} u_{\beta} + p(\mu) g_{\alpha\beta}
  \]
  
  future unit timelike \(u^\alpha\)

  mass-energy density \(\mu \geq 0\)

  isothermal pressure law \(p(\mu) = k^2 \mu\)

- **Gowdy symmetry on** \(T^3\)

  - The twist constants (Geroch, 1971; Chruściel, 1990) vanish
    \[
    \varepsilon_{\alpha\beta\gamma\delta} X^\alpha Y^\beta \nabla^\gamma X^\delta = \varepsilon_{\alpha\beta\gamma\delta} X^\alpha Y^\beta \nabla^\gamma Y^\delta = 0
    \]

  - Inhomogeneous cosmology: big bang / big crunch

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\(^3\)arXiv gr-qc:1004.0427  P. LeFloch and A. Rendall, A global foliation of Einstein-Euler spacetimes with Gowdy-symmetry
Areal coordinates Time function $t = R$: area of the $T^2$ orbits

$$g = e^{2(\nu - U)} \left( -dR^2 + \alpha^{-1} d\theta^2 \right) + e^{2U} \left( dx + A \, dy \right)^2 + e^{-2U} R^2 dy^2$$

with $\theta \in S^1$ and $R \in [R_0, R_1)$ (for some $R_0 > 0$) and $(x, y) \in T^2$

Finite energy condition

4 metric coefficients + fluid variables

$$U, A \in L^\infty_{\text{loc}}([R_0, R_1), W^{1,2}(S^1)), U_R, A_R \in L^\infty_{\text{loc}}([R_0, R_1), L^2(S^1))$$

$$\nu_R, \nu_\theta \in L^\infty_{\text{loc}}([R_0, R_1), L^1(S^1)), \alpha \in L^\infty_{\text{loc}}[R_0, R_1), L^\infty(S^1))$$

$$T_{\mu\nu}(\mu, u) \in L^\infty_{\text{loc}}([R_0, R_1), L^1(S^1))$$

Remark. Regularity stated first in conformal coordinates:

- $\nabla R \in W^{1,1}$ follows from Einstein's constraints
- $\nabla R$ timelike
- Choose the area $R$ as the time variable
FORMULATION OF THE INITIAL VALUE PROBLEM

Describing Gowdy symmetric initial data set on $T^3$

- **Geometry**
  - A Riemannian 3-manifold $(T^3, \bar{g})$ \(W^{1,2}\) regularity
  - A symmetric tensor field $\bar{k}$ \(L^2\) regularity

- **Matter**
  - Mass-energy density $\bar{\mu}$ \(L^1\) regularity
    (measured by an observer moving orthogonally to the foliation slices)
  - Fluid momentum $\bar{J}$ \(L^1\) regularity

- **Einstein constraints:**
  \[ R_{\bar{g}} = - (\text{tr} \, \bar{g})^2 + |\bar{k}|^2 + 2\bar{\mu} \]
  \[ \text{tr} (\nabla \bar{k}) - \nabla (\text{tr} \, \bar{k}) = \bar{J} \]

  - Understood in the distribution sense (derivation of \(L^p\) functions)
  - Additional (but limited) regularity implied

- **Invariance of these data under Gowdy symmetry** (Lie derivative understood in the distribution sense)
Weakly regular Gowdy symmetric $T^3$-spacetimes.

A foliation of the spacetime $\mathcal{M}$ by spacelike hypersurfaces $(\mathcal{M}_t \simeq T^3, g_t)$

- **Geometry**
  - Riemannian metric $g_t$ \[ \mathcal{W}^{1,2} \text{ regularity} \]
  - Second fundamental form $k_t$ \[ L^2 \text{ regularity} \]

- **Matter**
  - Mass-energy density $\mu$ \[ L^1 \text{ regularity} \]
  - Momentum vector $\mu u^\alpha$ \[ L^1 \text{ regularity} \]

- **Weak solution** to the Einstein-Euler equations
  - In the distribution sense
    \[ R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = T_{\alpha\beta} \]
  - Curvature defined in the distributional sense (derivatives of $L^p$ functions)
  - Implies the Euler equations in the weak sense (derivatives of $L^1$ functions)
    \[ \nabla^\alpha T_{\alpha\beta} = 0 \]
PROOF OF NONLINEAR STABILITY of Gowdy spacetimes with compressible matter included

Theorem. PLF & Rendall 2010

($\mathcal{M} \cong T^3, \bar{g}, \bar{k}, \bar{\mu}, \bar{J}$): finite energy, Gowdy symmetric, initial data set with constant area $R$

- **Existence theory**
  - Finite energy, Gowdy symmetric spacetime ($\mathcal{M}, g, \mu, \mu u^\alpha$)
  - A future Cauchy development of ($\mathcal{M}, \bar{g}, \bar{k}, \bar{\mu}, \bar{J}$), globally covered by a single coordinate chart

- **Global foliation** by spacelike hypersurfaces
  - A globally- and geometrically-defined time function $t$ coinciding with $\pm$ the area $R$
  - $R$ : two-dimensional spacelike orbits of the $T^2$ isometry group

- **$R$ increasing or decreasing toward the future**
  - Expanding spacetime $t = R \in [t_0, +\infty)$; $t_0 > 0$
    (hence the area grows without bound)
  - Contracting spacetime $t = -R \in [t_0, t_1)$; $t_0 < t_1 \leq 0$
Future Vanishing Area Property

Theorem. Grubic & PLF 2014

For future contracting Einstein-Euler spacetimes with Gowdy symmetry, provided some geometrically-defined invariant $D$ associated with the Gowdy symmetry is non-vanishing

- one has $t_1 = 0$
- Therefore, the area of the $T^2$ orbits of symmetry approaches 0 in the future.
- This condition is optimal within the class of spatially homogeneous spacetimes.

Motivated by earlier results for vacuum spacetimes

- Gowdy spacetimes admits a wave-map structure (Ringström 2004): the invariant $D$
- Crushing singularity property for vacuum $T^2$–symmetric spacetimes (Isenberg and Weaver)

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$^4$ArXiv qr-qc: 1411.3269 N. Grubic and P.G. LeFloch, On the area of the symmetry orbits in weakly regular Einstein-Euler spacetimes with Gowdy symmetry
Weak formulation in areal coordinates \( t = R \)

\[
g = e^{2(\nu-U)}(dR^2 + a^{-2}d\theta^2) + e^{2U}(dx + Ady)^2 + e^{-2U}R^2dy^2
\]

(1 + 1) nonlinear wave map structure in \((U, A)\), with the hyperbolic space as a target with metric

\[
dU^2 + e^{2U}dA^2
\]

Isometries of the hyperbolic space carry solutions to solutions: translations, dilations, inversions, reflections. By Noether theorem:

- In areal coordinates, coordinate-dependent invariants: \( \mathcal{A} \) (dilations), \( \mathcal{B} \) (translations), \( \mathcal{C} \).
- The combination \( \mathcal{D} := \mathcal{A}^2 + \mathcal{BC} \) of the invariants remains unchanged under an action of isometries.

- Ringstrom: time-asymptotics in the expanding direction and future causal geodesic completeness. A solution defines a closed loop in hyperbolic space (length, shape).
- Berger & Moncrief: numerical investigations
Applying translations, dilations, inversions, reflections in order to impose specific values for $A, B, C$.

$A_1, B_1$ and $C_1$ represent the conserved quantities of the transformed solution

**Lemma**

- When $D \equiv A^2 + BC > 0$, there is an isometry such that
  \[ A_1 = -\sqrt{D}, \quad B_1 = C_1 = 0 \]

- When $D = 0$, there is a transformation
  \[ A_1 = B_1 = 0 \quad \text{while} \quad C_1 = 0 \text{ or } C_1 = 1 \]

- When $D < 0$, it is obviously not possible to achieve $B_1 = 0$, however there exists a transformation
  \[ A_1 = 0, \quad B_1 = -1, \quad C_1 = |D| \]

**Spatially homogeneous case:**

- One always has $A^2 + BC \geq 0$
- In the special case $A^2 + BC = 0$, only $A = B = C = 0$ is possible
Spatially homogeneous spaces - Bianchi type I solutions (three-dim. Abelian group of isometries)

\[ \alpha = \frac{3k^2 + 1}{4} \]

\[
\left( a^{-1}t(U_t - 1/(2t)) \right)_t = \frac{e^{4U}}{2at} A_t^2
\]

\[
\left( a^{-1}t^{-1}e^{2U} A_t \right)_t = -2 \frac{e^{2U}}{at} U_t A_t
\]

\[ a_t = -ate^{2(\nu-U)} \mu(1 - k^2) \]

\[
\left( a^{-1}te^{2(\nu-U)} \mu \right)_t = a^{-1}te^{2(\nu-U)} \mu(1 - k^2) \left( -\frac{1}{(1-k^2)} \frac{\alpha}{t} + atE_1(t) \right)
\]

- Normalized density \( m := \frac{4t^2}{3} e^{2(\nu-U)} \mu \)

- New time variable

\[ \tau = -\log \left( \frac{t}{t_0} \right)^{1-\alpha} \]

\[ \tau \geq 0 \]
Proposition

(A) General case $\mathcal{D} > 0$: solution is defined for all $t \in [t_0, 0)$

- $a, tU_t, A_t$ remain globally bounded
- $m \to 0$ as $t \to 0$ 
  “matter does not matter”

(B) Exceptional cases $\mathcal{D} = 0$

(i) Small mass density $m_0 < 1$
- same as above

(ii) Critical regime $m_0 = 1$: defined for times, but
- $a(t)$ blows-up at $t = 0$
- The normalized density $m = 1$ remains constant
  “matter matters”

(iii) Large mass density $m_0 > 1$
- Only defined on some interval $[t_0, t_1)$ with $t_1 \in (t_0, 0)$
  - $a(t)$ blows-up at $t_1$
  - $m(t)$ blows-up at $t_1$
  - the Kretschmann scalar blows-up
  - null singularity: the hypersurface $t = t_1$ is null
BACK to the general set-up: **Euler equations** in the weak sense

\[\partial_t \left( a^{-1} te^{2(v-U)} \mu \frac{1 + k^2 v^2}{1 - v^2} \right) + \partial_\theta \left( te^{2(v-U)} \mu \frac{1 + k^2 v}{1 - v^2} \right) = a^{-1} te^{2(v-U)} \mu (1 - k^2) \Sigma_0\]

\[\partial_t \left( a^{-1} te^{2(v-U)} \mu \frac{1 + k^2 v}{1 - v^2} \right) + \partial_\theta \left( te^{2(v-U)} \mu \frac{k^2 + v^2}{1 - v^2} \right) = a^{-1} te^{2(v-U)} \mu (1 - k^2) \Sigma_1\]

\[\Sigma_0 := -\frac{k^2}{(1 - k^2)t} - U_t + t(U_t^2 + a^2 U_\theta^2) + \frac{e^4 U}{4t}(A_t^2 + a^2 A_\theta^2)\]

\[\Sigma_1 := -a U_\theta + 2t a U_t U_\theta + \frac{e^4 U}{2t} a A_t A_\theta\]

System of two genuinely nonlinear hyperbolic equations
Controlling the rescaled mass-energy density
Consider a weak solution to the Einstein-Euler system on some \([t_0, t_c)\).

- Derive an upper bound on \(a\), which solely assumes a lower bound of the perimeter \(P(t) := \int_{S^1} a^{-1}(t, \theta) \, d\theta\) (volume of the \(t\)-constant slices for the conformal metric)
- Control the variation in space of the \(L^1\) averaged mass density ("second" Euler equation)

Proposition

As long as \(P\) is uniformly bounded below (integral bound) by a constant \(\kappa > 0\) on the open interval \([t_0, t_c) \subset (-\infty, 0)\), then one can find a function \(C_\kappa = C_\kappa(t) > 0\) such that

\[
\sup_{S^1} \int_{t_0}^t \frac{e^{2(\nu-U)} \mu}{1-\nu^2} (\tau, \cdot) \, d\tau \leq C_\kappa(t), \quad t \in [t_0, t_c)
\]

Consequently, thanks to

\[
a_t = -at e^{2(\nu-U)} \mu (1 - k^2),
\]

one in fact has a uniform bound on the \(L^\infty\) norm \(\sup_{[t_0, t_c] \times S^1} a\) and one can continue the solution beyond \(t_c\).
Controlling the conformal volume of the spacelike slices

Next, as long as \(|t| > 0\), the function \(\int_{S^1} a^{-1} d\theta\) remains strictly positive:

**Proposition**

Suppose the invariant \(D = A^2 + BC \neq 0\) is non-vanishing. Then, on any interval \([t_0, t_c) \subset (-\infty, 0)\)

- there exists a smooth function \(f = f(s)\) (involving polynomial and exponential factors and vanishing at the origin) such that

\[
\frac{P(t)}{t^2} = \frac{1}{t^2} \int_{S^1} a^{-1}(t, \theta) d\theta \geq f\left(\frac{|D|}{t^2}\right)
\]

- Therefore, the perimeter can not approach zero (unless possibly \(t_c = 0\) and \(t \to t_c\)).

The proof distinguishes between \(D > 0\) and \(D < 0\), and the values of \(A, B, C\).
3. Stability of Khan-Penrose colliding spacetimes

Irrotational null fluids in plane symmetry\(^5\)

- Polarized Gowdy symmetry (two orthogonal Killing fields with vanishing twist) and an irrotational null fluid with potential \(\psi\)
- Two null hypersurfaces \(\mathcal{N} \cup \overline{\mathcal{N}}\) intersecting along a spacelike two-plane \(\mathcal{P} := \mathcal{N} \cap \overline{\mathcal{N}}\)

Characteristic initial value problem with \(W^{1,2}\) initial data

- Incoming radiation
- Weak regularity singular curvature
- No compatibility condition assumed at the intersection

Remarks.

- Many works on the characteristic initial value problem (no symmetry, smooth data): Friedrich, Stewart, Dossa, Cagnac, Rendall, Christodoulou, Choquet-Bruhat, Chruściel
- Generic formation of trapped surfaces / short pulse method: Christodoulou, Klainerman-Rodnianski, etc.
- Luk, Rodnianski, Dafermos, X. An localized singularities

\(^5\)ArXiv gr-qc:1004.2343 P. LeFloch and J.M. Stewart, The characteristic initial value problem for plane symmetric spacetimes with weak regularity
Definition

An initial data with weak regularity (assuming plane symmetry throughout): \((\mathcal{N}, e^a dU dy dz), (\overline{\mathcal{N}}, e^{\overline{a}} dV dy dz)\) with boundaries identified along a two-plane \(\mathcal{P}\)

\[\mathcal{N} := \{ U \geq U_0 \}, \quad \overline{\mathcal{N}} := \{ V \geq V_0 \}, \quad \mathcal{P} := \{ U = U_0, V = V_0 \}\]

- \(a, \overline{a}\) are belong to \(W^{1,1}\) and are normalized so that \(a|_\mathcal{P} = \overline{a}|_\mathcal{P} = 0\):

\[
\int_{\mathcal{N}} (|a| + |\partial_U a|) \, e^a \, dU + \int_{\overline{\mathcal{N}}} (|\overline{a}| + |\partial_V \overline{a}|) \, e^{\overline{a}} \, dV < +\infty
\]

- NP Ricci scalars \(\Phi_{00}\) and \(\Phi_{22}\) prescribed on the hypersurfaces \(\mathcal{N}\) and \(\overline{\mathcal{N}}\), respectively, with \(0 \leq \Phi_{00} \in L^1(\mathcal{N})\) and \(0 \leq \Phi_{22} \in L^1(\overline{\mathcal{N}})\):

\[
\int_{\mathcal{N}} \Phi_{00} \, e^a \, dU + \int_{\overline{\mathcal{N}}} \overline{\Phi}_{22} \, e^{\overline{a}} \, dV < +\infty
\]

- NP Weyl scalars \(\Psi_0 \in W^{-1,2}(\mathcal{N})\) and \(\overline{\Psi}_4 \in W^{-1,2}(\overline{\mathcal{N}})\)

- Connection NP scalars \(\mu_0, \sigma_0, \lambda_0, \mu_0\) on \(\mathcal{P}\)
Theorem. Weakly regular plane-symmetric colliding spacetimes

PLF & Stewart 2010

1. There exists a $W^{1,2}$ weakly regular, plane-symmetric Einstein-Euler development $(\mathcal{M}, g, \psi)$ of the initial data set

$$(a, \psi_0, \Phi_{00}) = (a, \psi_0, \Phi_{00}) \quad \text{on the null hypersurface } \mathcal{N}$$

$$(a, \psi_4, \Phi_{22}) = (\bar{a}, \bar{\psi}_4, \bar{\Phi}_{22}) \quad \text{on the null hypersurface } \overline{\mathcal{N}}$$

$$(\mu, \sigma, \lambda, \mu) = (\mu_0, \sigma_0, \lambda_0, \mu_0) \quad \text{on } \mathcal{P}$$

which has past boundary

$$\{U_0 > U > U_0; V = V_0\} \cup \{U = U_0; \overline{V}_0 > V > V_0\} \subset \mathcal{N} \cup \overline{\mathcal{N}}$$

2. For generic initial data, the curvature blows up to infinity (in the $H^{-1}$ norm) and makes no sense even as a distribution, as one approaches its future boundary:

$$\mathcal{B}_0 := \{F(U) + G(V) = 0\}$$

determined by some functions $F, G$ in $W^{1,2}$.
A version of Penrose strong’s censorship conjecture within the class of plane-symmetric spacetimes with weak regularity

Generically, the spacetime is inextendible beyond $B_0$ within the class of $W^{1,2}$ regular spacetimes

- Holds for generic initial data
  arbitrary data can always be perturbed in the energy norm, so that the perturbed initial data generate a spacetime whose curvature blows-up on $B_0$.
- Modulo a conformal transformation, $e^a$ and $e^{\bar{a}}$ could be chosen to be identically 1 on the initial hypersurface.
- Only two genuine degrees of freedom on each initial hypersurface
ELEMENTS OF PROOF
Reduction of null irrotational fluid

- Null fluid: pressure equal to its mass-energy density $p = \mu$

$$T_{\alpha\beta} = 2\mu u^\alpha u^\beta + \mu g_{\alpha\beta}$$

sound speed = light speed.

- Second contracted Bianchi identities: Euler equations

$$(u^\alpha \nabla_\alpha \mu) u^\beta + \mu (\nabla_\alpha u^\alpha) u^\beta + \mu u^\alpha \nabla_\alpha u^\beta - \frac{1}{2} \nabla^\beta \mu = 0$$

- Irrotational fluid: $u^\alpha$ is a (normalized) gradient of a scalar potential $\psi$:

$$u^\alpha = \frac{\nabla^\alpha \psi}{\sqrt{-\nabla_\beta \psi \nabla^\beta \psi}}$$

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Rodnianski and Speck without symmetry for smooth data, 2012
Multiply by $u_\beta$ and obtain

$$2\nabla_\alpha \mu u^\alpha - 2\mu \nabla_\alpha u^\alpha + 2\mu u^\alpha u_\beta \nabla_\alpha u^\beta - \nabla_\alpha \mu u^\alpha = 0,$$

which, in view of $u_\beta \nabla_\alpha u^\beta = 0$, simplifies

$$u^\alpha \nabla_\alpha \mu + 2\mu \nabla_\alpha u^\alpha = 0.$$

With $\Sigma = \frac{1}{2} \log \mu$

$$u^\alpha \nabla_\alpha \Sigma + \nabla_\alpha u^\alpha = 0$$

Multiply by the projection $H_{\beta\gamma} = g_{\beta\gamma} - u_\beta u_\gamma$ and obtain

$$H^{\alpha\gamma} \nabla_\alpha \mu - 2\mu u^\alpha \nabla_\alpha u^\gamma = 0$$

therefore

$$H^{\alpha\gamma} \nabla_\alpha \Sigma - u^\alpha \nabla_\alpha u^\gamma = 0$$

We now impose for some $\psi$

$$u_\alpha = \frac{\nabla_\alpha \psi}{\sqrt{-\nabla_\beta \psi \nabla^\beta \psi}}$$
Reduced equations for irrotational null fluids

\[
\begin{align*}
\nabla_\alpha \left( \frac{\mu^{1/2}}{\sqrt{-\nabla_\beta \psi \nabla^\beta \psi}} \nabla_\alpha \psi \right) &= 0 \\
\nabla_\beta \left( \Sigma - \log \sqrt{-\nabla_\alpha \psi \nabla^\alpha \psi} \right) &= k \nabla_\beta \psi
\end{align*}
\]

with \( k = \frac{\nabla_\alpha \psi \nabla_\alpha \Sigma}{\nabla_\alpha \psi \nabla_\alpha \psi} - \frac{\nabla_\alpha \psi \nabla_\beta \psi \nabla_{\alpha \beta} \psi}{(\nabla_\alpha \psi \nabla_\alpha \psi)^2} \)

- The gradient of \( \Sigma - \log \sqrt{-\nabla_\alpha \psi \nabla^\alpha \psi} \) is parallel to \( \nabla \psi \), so that the former can be expressed as a function \( F(\psi) \) for some \( F \).
- By replacing our choice of \( \psi \) by some \( G(\psi) \), we can always arrange that \( \Sigma - \log \sqrt{-\nabla_\alpha \psi \nabla^\alpha \psi} = 0. \)

\[
\mu = -\nabla_\alpha \psi \nabla^\alpha \psi
\]

- **Relativistic analogue of Bernoulli’s law**
  - for irrotational flows in classical fluid mechanics
  - determines \( \mu \) algebraically, once we know the velocity

- **Wave equation on a curved space** for the potential

\[
\Box_g \psi = 0
\]
Reduction of the collision problem In double null coordinates

- \( g = e^{2a} \, dudv - e^{2b} \left( e^{2c} \, dy^2 + e^{-2c} \, dz^2 \right) \), unknowns \( a, b, c \) and \( \psi \)
- Einstein equations: choice of null coordinates so that the future boundary is \( u + v = 0 \)
- Essential Einstein-Euler equations: Euler-Poisson-Darboux equation

\[ \Box_g \psi = 0 = \Box_g c, \quad \psi_{uv} + \frac{C(u,v)}{u+v}(\psi_u + \psi_v) = 0 \]

- Associated Riemann function (\( F \): hypergeometric function)

\[ \varphi(u', v'; u, v) = \left( \frac{u' + v'}{u' + v} \right)^{1/2} \left( \frac{u' + v'}{u + v'} \right)^{1/2} F \left( \frac{1}{2}, \frac{1}{2}; 1; \frac{(v' - v)(u' - u)}{(v' + u)(u' + v)} \right) \]

- Representation formula for the Goursat problem

\[ \psi(u, v) = \varphi(u_0, v_0; u, v) \psi(u_0, v_0) + \int_{u_0}^{u} \varphi(u', v_0; u, v) B[\psi](u', v_0) \, du' \]

\[ + \int_{v_0}^{v} \varphi(u_0, v'; u, v) \overline{B}[\psi](u_0, v') \, dv' \]

where the characteristic data yield \( \psi(u_0, v_0) \) and

\[ B[\psi](u', v_0) := \psi_{u'}(u', v_0) + \frac{1}{2} (u' + v_0)^{-1} \psi(u', v_0) \]

\[ \overline{B}[\psi](u_0, v') := \psi_{v'}(u_0, v') + \frac{1}{2} (u_0 + v')^{-1} \psi(u_0, v') \]
Analysis of the blow-up near the singularity

- Study the singularities of the Riemann function
- Behavior on the future boundary $u + v = 0$

$$W^{1,2} \ni \psi(u, v) \sim \Psi(u) \log |u + v|$$

for some $\Psi = \Psi(u) \in L^2$ determined from the initial data

- Metric $g = e^{2a} \, du \, dv - e^{2b} \left( e^{2c} \, dy^2 + e^{-2c} \, dz^2 \right)$

**Lemma**

Leading terms for the metric

$$a(u, v) \sim A(u - v) \log |u + v|, \quad A = C^2 + \frac{1}{2} \Psi^2 - \frac{1}{4}$$

$$c(u, v) \sim C(u - v) \log |u + v|, \quad A \in L^1, \quad C \in L^2, \quad \Psi \in L^2$$

with explicit (but extremely tedious) formula for $A, C, \Psi$. Therefore, at least formally

$$g \sim |u + v|^{2A(u-v)} \, du \, dv - |u + v|^{1+2C(u-v)} \, dy^2 + |u + v|^{1-2C(u-v)} \, dz^2$$
Proposition

Leading terms in the Ricci curvature as $|u + v| \to 0$:

$$R \lesssim -\Psi^2 |u + v|^{-2 - 2A} \lesssim -\Psi^2 |u + v|^{-3/2} \leq 0$$

$$|R_{\alpha\beta} R^{\alpha\beta}|^{1/2} \approx |\Phi_{00} \Phi_{22} + 18 \Lambda^2 + 2 \Phi_{11}^2 - 4 |\Phi_{01}|^2 + |\Phi_{02}|^2|^{1/2}$$

$$\approx \Psi^2 \left(1 + |u + v|^{-4A}\right)^{1/2} |u + v|^{-2}$$

$$\gtrsim \Psi^2 |u + v|^{-2}$$

- $\Phi_{00}, \Phi_{01}, \Phi_{11}, \Phi_{22}$ belong to $L^1$
- $R$ and $(R_{\alpha\beta} R^{\alpha\beta})^{1/2}$ belong to $L^1$
- Non-blow-up would require (at least) that $\Psi = 0$

explicit condition in the initial data excludes only non-generic initial data
by $W^{1,2}$ perturbing the incoming matter radiation, one can always ensures that $\Psi$ is non-vanishing.
4. Formation of trapped surfaces in spherical symmetry

Gravitational collapse of compressible matter with shocks

Theorem. Burtscher & PLF 2014

By solving the initial value problem from a large class (specified explicitly below by a short pulse Ansatz) of spherically-symmetric initial data sets \((\mathcal{H}, \bar{g}, \bar{\rho}, \bar{j})\) we generate a spherically symmetric, Einstein-Euler future development \((\mathcal{M}, g, \mu, u)\) possibly containing shocks such that:

2. The initial hypersurface does not contain trapped spheres.
3. The development contains trapped spheres.

- **Eddington-Finkelstein coordinates**
  \[
g = -ab^2 \, dv^2 + 2b \, dvdr + r^2 \, g_{S^2}\]
  with \(v \in [v_0, v_*], \ r \in [0, r_0]\)

- \(b = b(v, r) > 0\) but \(a = a(v, r)\) may change sign

- Regularity at the center: \(\lim_{r \to 0} (a, b)(v, r) = (1, 1)\)

- Schwarzschild solution \(g = -(1 - 2m/r) \, dv^2 + 2dvdr + r^2 \, g_{S^2}\)

---

\(^7\)ArXiv gr-qc: 1411.3008  A. Burtscher and P. LeFloch, The formation of trapped surfaces in spherically-symmetric Einstein-Euler spacetimes
Essential Euler equations understood in the sense of distributions

\[ 0 = \partial_v \left( \mu (1 + k^2) u^0 u^0 \right) + \partial_r \left( \mu (1 + k^2) u^0 u^1 + k^2 \frac{\mu}{b} \right) \]

\[ + \left( \frac{2b_v}{b} + \frac{a_r b}{2} + ab_r \right) \mu (1 + k^2) u^0 u^0 \]

\[ + \left( \frac{b_r}{b} + \frac{2}{r} \right) \left( \mu (1 + k^2) u^0 u^1 + k^2 \frac{\mu}{b} \right) - \frac{2k^2}{rb} \mu \]

\[ 0 = \partial_v \left( \mu (1 + k^2) u^0 u^1 + k^2 \frac{\mu}{b} \right) + \partial_r \left( \mu (1 + k^2) u^1 u^1 + k^2 \mu a \right) \]

\[ + \left( -\frac{a_v b}{2} + \frac{aa_r b^2}{2} + a^2 b b_r \right) \mu (1 + k^2) u^0 u^0 \]

\[ + \left( \frac{b_v}{b} - a_r b - 2ab_r \right) \left( \mu (1 + k^2) u^0 u^1 + k^2 \frac{\mu}{b} \right) \]

\[ + \left( \frac{2b_r}{b} + \frac{2}{r} \right) \left( \mu (1 + k^2) u^1 u^1 + k^2 \mu a \right) - \frac{2k^2 a}{r} \mu \]

Remark. Earlier works on “local” existence

- going back to Glimm (1965), Groah-Temple (2004), LeFloch and Stewart (2005)

- here, we prove a “semi-global” result (well beyond “local” existence) in order for the trapped surfaces to form
Normalized mass-energy and normalized velocity

\[ u^1 = \frac{1}{2} \left( abu^0 - \frac{1}{bu^0} \right) \]

\[ M := b^2 \mu u^0 u^0 \in (0, +\infty), \quad V := \frac{u^1}{bu^0} - \frac{a}{2} \in (-\infty, 0) \]

System of two nonlinear hyperbolic equations

\[ K^2 := \frac{1 - k^2}{1 + k^2} \in (0, 1) \]

\[ \partial_v U + \partial_r F(U, a, b) = S(U, a, b) \]

\[ U := M \left( \frac{a}{2} + K^2 V \right), \quad F(U, a, b) := bM \left( \frac{a}{2} + K^2 V \right) \]

\[ S(U, a, b) := \left( \begin{array}{c}
S_1(M, V, a, b) \\
S_2(M, V, a, b)
\end{array} \right), \quad S_1(M, V, a, b) := -\frac{1}{2r} bM (1 + a + 4V) \]

\[ S_2(M, V, a, b) := -\frac{1}{2r} bM \left( a^2 + 2aV(2 + K^2) - 2K^2 V + 4V^2 \right) - 16\pi (1 - K^2) rb M^2 V^2 \]

\[ a(v, r) = 1 - \frac{4\pi (1 + k^2)}{r} \int_0^r \frac{b(v, r')}{b(v, r)} M(v, r') (2K^2 |V(v, r')| + 1) r'^2 dr' \]

\[ b(v, r) = \exp \left( 4\pi (1 + k^2) \int_0^r M(v, r') r' dr' \right) \]
Bounded Variation (BV) solutions

- hyperbolic, genuine nonlinear
- formation of shocks in finite time

Definition

A spherically symmetric, Einstein-Euler spacetime with bounded variation in Eddington-Finkelstein coordinates

\[
g = -ab^2 \, dv^2 + 2b \, dvdr + r^2 \, (d\theta^2 + \sin^2 \theta \, d\varphi^2)
\]

with \( v \in I := [v_0, v_*] \), \( r \in J := [0, r_0) \) is, by definition, a weak solution to the Einstein-Euler system with the following regularity:

- Normalized mass density and velocity

\[
M, \, V \in L^\infty(I, BV(J)) \cap \text{Lip}(I, L^1(J))
\]

- Metric coefficients \( a, b \)

\[
a_v, \, ra_r, \, b_r \in L^\infty(I, BV(J)) \cap \text{Lip}(I, L^1(J))
\]
Analysis of static Einstein-Euler equations

Lemma

System in terms of the local Hawking mass $m$ and the fluid density $\mu$

\[
m_r = 4\pi r^2 \mu
\]
\[
\mu_r = -\frac{(1 + k^2)\mu}{r - 2m} \left(4\pi r^2 \mu + \frac{m}{rk^2}\right)
\]

The remaining unknowns $V, M, a, b$ are recovered from $(m, \mu)$ by

\[
a^2 = 4V^2
\]
\[
V = -\frac{a}{2} = \frac{m}{r} - \frac{1}{2}
\]
\[
aM = \left(1 - \frac{2m}{r}\right) M = \mu
\]
\[
b(r) = \exp \left(4\pi(1 + k^2) \int_0^r \frac{r'^2 \mu(r')}{r' - 2m(r')} \, dr'\right)
\]
Theorem

Given any mass-energy density $\mu_0 > 0$ at the center $r = 0$, there exists a unique global solution $(m, \mu)$ to the static Einstein-Euler system in Eddington-Finkelstein coordinates with prescribed values at the center

$$\lim_{r \to 0} m(r) = 0, \quad \lim_{r \to 0} \mu(r) = \mu_0.$$ 

Moreover, $m, \mu$ are smooth and positive on $(0, +\infty)$ with

$$\lim_{r \to +\infty} \mu(r) = 0.$$

Remark: Rendall-Schmidt (1991), Ramming and Rein (2013)

Trapped surface formation: the class of initial data

- Short-pulse: compactly-supported perturbation of a static solution
- Initially localized on an interval $[r_* - \delta, r_* + \delta]$ for some sufficiently small $\delta$
Riemann problem for the evolution of discontinuities

- Generalized Riemann problem

\[ \partial_v U + \partial_r F(U, a, b) = S(U, a, b) \]

\[ (M, V)(v_0, r) = \begin{cases} (M_-, V_-), & r < r_0 \\ (M_+, V_+), & r_0 < r \end{cases} \]

- Blow-up analysis near the initial discontinuity: geometrical effects “suppressed”

- Fundamental structures: shock waves, rarefaction waves

Proposition

The (classical) Riemann problem associated with the Euler system on a uniform Eddington-Finkelstein background and with arbitrary Riemann initial data \( U_L, U_R \)

- admits a unique self-similar solution \( U = U(r/v) \)
- made of two elementary waves
- each of them being a rarefaction wave or a shock wave
Riemann invariants $w, z$

**Rarefaction curves**

\[
\begin{align*}
R_1^\rightarrow(U_L) &= \left\{ (w, z) \mid w(M, V) = w(M_L, V_L) \text{ and } z(M, V) \leq z(M_L, V_L) \right\} \\
R_2^\leftarrow(U_R) &= \left\{ (w, z) \mid w(M, V) \geq w(M_R, V_R) \text{ and } z(M, V) = z(M_R, V_R) \right\}
\end{align*}
\]

**1-shock curve** issuing from a state left-hand state $U_L$

\[
\begin{align*}
S_1^\rightarrow(U_L) &= \left\{ M = M_L \Phi_-(V/V_L); \quad V/V_L \in [1, \infty) \right\} \\
s_1(U_L, U) &= \Sigma_+(V_L, V/V_L)
\end{align*}
\]

**2-shock curve** issuing from a right-hand state $U_R$

\[
\begin{align*}
S_2^\leftarrow(U_R) &= \left\{ M = M_R \Phi_+(V/V_R); \quad V/V_R \in (0, 1] \right\} \\
s_2(U, U_R) &= \Sigma_-(V_R, V/V_R)
\end{align*}
\]
Invariant regions

- Invariant region principle for the Riemann invariants:

\[ \Omega_\mu := \{(w, z) | -\delta \leq w, z \leq \delta\} \]

If the data \( U_L, U_R \) belong to \( \Omega_\delta \) for some \( \delta > 0 \), then so does the Riemann solution.

- Positivity \( M > 0 \) and condition \( V < 0 \) preserved by the Riemann problem

Random-choice method

- Sequence of approximate solutions
  - Discretization of a general initial data set by a piecewise constant data with finitely many jumps
  - Solve the generalized Riemann problem at each initial discontinuity

- Marching in forward time directions with discrete time levels
  - randomly choose of a state within the local Riemann problem (equidistributed sequences)
  - construction by induction in time

- Compactness
  - Derive a bound on the total variation of \( \log \mu \)
  - Bounds that are independent of the discretization parameter
Objective
Existence on a sufficiently long interval \([v_0, v_\ast]\) so that trapped surfaces do form in the future

- Localized perturbation of a static solution

\[
M = M^{(0)} + M^{(1)}, \quad V = V^{(0)} + V^{(1)}
\]
\[
a = a^{(0)} + a^{(1)}, \quad b = b^{(0)} + b^{(1)}
\]

- Initial data untrapped: \(a(v_0, \cdot) > 0\)

- Choose \(a_v\) initially large (negative) within a small interval \([r_\ast - \delta, r_\ast + \delta]\) (short pulse)

- In view of

\[
a_v = 2\pi rbM(1 + k^2)(a^2 - 4V^2)
\]

we choose \(-V > 0\) to be initially very large on a small interval (a pulse)

Challenge: we need to carefully monitor the evolution of \(a\)!
The short pulse data
We choose a radius $r_\ast > 0$, a region of perturbation $[r_\ast - \delta, r_\ast + \delta]$ given by a small $\delta > 0$

- The pulse added to be the velocity is a step function

$$V_0^{(1)}(r) := \begin{cases} 
0, & r < r_\ast - \delta \\
\frac{V^{(0)}(r)}{h}, & r \in [r_\ast - \delta, r_\ast + \delta] \\
0, & r > r_\ast + \delta
\end{cases}$$

with a scale $h = h(r_\ast, \delta)$

- No perturbation for the mass density

$$M_0^{(1)} = 0, \quad b_0^{(1)} = 0$$

- It follows that

$$a_0(r) = 1 - \frac{4\pi(1 + k^2)}{r} \int_0^r \frac{b^{(0)}(s)}{b^{(0)}(r)} M^{(0)}(s) \left(1 + K^2 \left(1 + \frac{1}{h} \chi_{[r_\ast - \delta, r_\ast + \delta]} \right) a^{(0)}(s) \right)s^2 ds$$

with $\chi_{[r_\ast - \delta, r_\ast + \delta]}$ the characteristic function
Lemma

Given $r_* > \Delta > 0$, there exist constants $C_1, C_2, C_3 > 0$ depending on $r_*$ and $\Delta$ such that

for all $\delta, h > 0$ with $\frac{\delta}{h} \leq \frac{1}{C_1}$

$$0 < a_0(r) \leq a^{(0)}(r), \quad r \in [0, r_* + \Delta]$$

$$a_v(v_0, r) \begin{cases} 
= 0, & r \in [0, r_* - \delta) \\
\leq -C_2 \frac{\delta}{h^3}, & r \in [r_* - \delta, r_* + \delta] \\
\leq -C_3 \frac{\delta}{h}, & r \in (r_* + \delta, r_* + \Delta]
\end{cases}$$
An approximate solution $M_\#$, $V_\#$, $a_\#$, $b_\#$ to the Euler–Einstein system is said to satisfy the **short-pulse property** if

- there exist constants $C_0$, $C$, $C_b$, $\Lambda > 0$ (depending only on the chosen static solution)
- an exponent $\kappa > 1$ (depending on the sound speed $k$),
- such that for all $v \in [v_0, v_*]$ with $v_* := v_0 + \tau h^\kappa$ (the time of existence):

  - **In the domain of influence of the initial pulse**
    
    \[
    \frac{1}{C_0} e^{-C \frac{v - v_0}{h^\kappa}} \left(1 + \frac{1}{h}\right)^{-\kappa_0} \leq M_\#(v, r) \leq C_0 e^{C \frac{v - v_0}{h^\kappa}} \left(1 + \frac{1}{h}\right)^{\kappa_0}
    \]
    
    \[
    \frac{1}{C_0} e^{-C \frac{v - v_0}{h^\kappa}} \leq -V_\#(v, r) \leq C_0 e^{C \frac{v - v_0}{h^\kappa}} \left(1 + \frac{1}{h}\right)
    \]

  - **In the domain of dependence of the initial pulse** (matter density bounded below and velocity very large negative)

    \[
    \frac{1}{C_0} e^{-C \frac{v - v_0}{h^\kappa}} \leq M_\#(v, r) \leq C_0 e^{C \frac{v - v_0}{h^\kappa}}
    \]
    
    \[
    \frac{1}{C_0} e^{-C \frac{v - v_0}{h^\kappa}} \left(1 + \frac{1}{h}\right) \leq -V_\#(v, r) \leq C_0 e^{C \frac{v - v_0}{h^\kappa}} \left(1 + \frac{1}{h}\right)
    \]
Theorem. Burtscher & LeFloch 2014

- Fix $k \in (0, 1)$ and $\kappa \in [\kappa(k), 2)$. Given any $\mu_0 > 0$, let $M^{(0)}, V^{(0)}$ be the static solution with density $\mu_0$ at the center.

- Fix any $r_*> \Delta > 0$ together with perturbation parameters $h, \delta > 0$ satisfying $\delta \leq \frac{h}{C_1}$ with $C_1$ as above. Let $Z_0 = (M_0, V_0, a_0, b_0)$ be a short pulse perturbation of the static solution.

Then there exists $\tau > 0$ so that the approximate solutions $Z_\#$ constructed above are well-defined on the time interval $[v_0, v_*]$ with $v_* = v_0 + \tau h^\kappa$:

- **Short-pulse property**

- **Uniform BV property**
  
  $$\sup_{v \in [v_0, v_*]} TV (Z_\#(v, \cdot) - Z^{(0)}) \lesssim TV (Z_0 - Z^{(0)})$$

- **Lipschitz continuity property** in the $L^1$ norm ($v, v' \in [v_0, v_*]$)
  
  $$\int_{R_+^+(v)}^{R_-^-(v)} |U_\#(v, r) - U_\#(v', r)| \, dr \lesssim TV (U_0 - U^{(0)}) (|v - v'| + \Delta v)$$

The sequence $Z_\#$ sub-converges pointwise to a limit $Z = (M, V, a, b)$:

- bounded variation solution to the Euler–Einstein system

- satisfying the initial condition without trapped surfaces

- trapped surface formation in the future
Introduction to mathematical general relativity
September 14th to 18th

Recent advances in general relativity
September 23rd to 25th

Geometric aspects of mathematical relativity
September 28th to October 1st - Montpellier

Dynamics of self-gravitating matter
October 26th to 29th

General relativity: a celebration of the 100th anniversary
November 16th to 20th

Relativity and geometry - in memory of A. Lichnerowicz
December 14th to 16th