Integrability, Solvability and Enumeration.

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Many 2d lattice models are solvable for some properties and/or some lattices.

Why this is so is not fully understood.

There are various numerical techniques that, magically, seem to be exact for the solvable situations and not for the others.

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THE TWO-DIMENSIONAL ISING MODEL

- Take $t_1 = \tanh(J_x/kT)$ and $t_2 = \tanh(J_y/kT)$ in directions $x$, $y$.
- The log of the reduced p.f. is
  \[
  \log \Lambda(t_1, t_2) = \sum_{n,m} a_{n,m} t_1^{2m} t_2^{2n} = \sum_n R_n(t_1^2) t_2^{2n}.
  \]
- Baxter showed that $R_n(t_1^2) = P_{2n-1}(t_1^2)/(1 - t_1^2)^{2n-1}$.
- $R_n$ rational, with num. and den. pols of degree $2n - 1$.
- The only singularity in the complex $t_1^2$ plane is at $t_1^2 = 1$.
- Maillard found an inversion relation for the p.f.,
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  \log \Lambda(t_1, t_2) + \log \Lambda(1/t_1, -t_2) = \log(1 - t_2^2).
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- There is also the obvious symmetry relation
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Remarkably, these two relations, plus the structure of $R_n$ suffices to determine, order by order, the numerator polynomials.

Alternatively, the two functional relations, and the structure of $R_n$ implicitly gives the Onsager solution.

A mere 70 years after Onsager, we could conjecture the exact solution from simple calculations—that of the first few $R_n$s.

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2d Ising susceptibility

- \( \chi(t_1, t_2) = \sum_{n,m} c_{n,m} t_1^m t_2^n = \sum_n H_n(t_1)t_2^n. \)

- The corresponding inversion and symm. relations are

  \[ \chi(t_1, t_2) + \chi(1/t_1, -t_2) = 0, \quad \chi(t_1, t_2) = \chi(t_2, t_1). \]

- The first few denominators of \( H_n(t_1) \) are:

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  \begin{align*}
  D_0(x) &= (1 - t_1) \\
  D_1(x) &= (1 - t_1)^2 \\
  D_2(x) &= (1 - t_1)^3(1 + t_1) \\
  D_3(x) &= (1 - t_1)^4 \\
  D_4(x) &= (1 - t_1)^4(1 + t_1)^3(1 - t_1^3) \\
  D_5(x) &= (1 - t_1)^6(1 + t_1)^2 \\
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The numerators are the same degree as denoms, and symmetric, unimodal with positive coefficients.

But the degree of the polynomials increases non-linearly.

The functional relations are insufficient to determine the numerator.

In the famous paper by Wu, McCoy, Tracy and Barouch, \( \chi(t) = \sum \chi^{(2n+1)}(t) \), where \( \chi^{(2n+1)}(t) = O(t^{(2n+1)^2-1}) \).

\( H_4(t) \) sees the first occurrence of \((1 - t^3)\) in the denominator, and reflects the \( O(t^8) \) term that enters with \( \chi^{(3)} \).

Similarly, \( H_{12}(t) \) sees the first occurrence of \((1 - t^5)\) in the denominator, and reflects the \( O(t^{24}) \) term that enters with \( \chi^{(5)} \).
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\(H_n(t)\) is rational, with poles on the unit circle in the \(t\)-plane.

- These become dense as \(n \to \infty\).
- Then (barring miraculous cancellation) \(\chi(t_1, t_2)\) as a function of \(t_1\) for \(t_2\) fixed (a) has a natural boundary, and (b) is neither algebraic nor D-finite, despite the fact that \(H_n(t_1)\) is rational.

- Some models can be refined into a proof (absence of cancellations).
- If we could prove positivity and unimodality, that would do. (No cancellations then possible).
- Andrew Rechnitzer did this for SAPs, bond animals, bond trees.
- Absent a proof, a powerful tool to conjecture non-D-finiteness.
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Numerical tests for functions with natural boundaries usually involve forming Padé approximants.

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● Square, triangular, hexagonal critical manifold known.

● E.g. $v^3 + 3v^2 - q = 0$ for triangular, $v = e^K - 1$.

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$$v^6 + 6v^5 + 9v^4 - 2qv^3 - 12qv^2 - 6q^2v - q^3 = 0.$$ 

● Correct for $q = 2$ (Ising)

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Integrability, Solvability and Enumeration.

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Integrability, Solvability and Enumeration. Tony Guttmann
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- Defines a polynomial $P_B(q, v)$ whose zeros give the p.b.
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THREE-TERMINAL LATTICES: SQ, TRI AND HEX.

(Fig. from Jac-Scull). All interactions in up-pointing triangles.
(Fig. from Jac-Scull). All possible interactions between spins in triangles.

Boltzmann weight

$$w_{123} = c_0 + c_1 \delta_{23} + c_2 \delta_{13} + c_3 \delta_{12} + c_4 \delta_{123}.$$ 

Let $G_A = (V, A)$ be a sub-graph of $G$, a piece of the lattice. The p.f. is $Z = \sum_\sigma \exp(-\beta H) = \sum_{A \subseteq E} v^{|A|} q^{k(A)}$. (F-K)

$|A|$ is # of edges in $A$, and $k(A)$ is the # of conn. comps. of $G_A$. 

Integrability, Solvability and Enumeration.
THREE-TERMINAL LATTICES – CONTINUED.

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Three-terminal lattices – continued.

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On three-terminal lattices this becomes

\[ Z = \sum_{A \subseteq E} q^{k(A)} \prod_{p=0}^{4} (c_p)^{N_p}, \]

where \( N_p \) is the # of up-triangles of type \( c_p \).

- At criticality, the model is invariant under a rotation of \( \pi/3 \).
- This implies (Wu & Lin, 1980) \( c_4 = qc_0 \).
- Apply this to triang. lattice with arbitrary, inhom. two-spin interactions within up-pointing triangles, so \( c_0 = 1 \), \( c_i = v_i \), \( i = 1, 2, 3 \), and \( c_4 = v_1v_2v_3 + v_1v_2 + v_2v_3 + v_1v_3 \), then

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(Fig. from Jac-Scull). A $4 \times 4$ square basis. For the kagome and other unsolved cases a four-terminal lattice is needed. The calculation of $P_B(q, v)$ is much more complicated.
Jacobsen and Scullard initially gave a contraction-deletion method, but later give a probabilistic, geometric interpretation. Consider two copies of the basis separated by an arbitrary distance. If connected, we say there is an infinite 2D cluster. Denote the weight of this event as $W(2D; B)$. If not, there are no infinite clusters. This has weight $W(0D; B)$. Then, remarkably,

$$P_B(q, \{v\}) = W(2D; B) - qW(0D; B).$$
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(Fig. from Jac-Scull). Square lattice with checkerboard coupling.

Here, $W(2D; B) = v_1 v_2 v_3 v_4 + v_1 v_2 v_3 + v_1 v_2 v_4 + v_1 v_3 v_4 + v_2 v_3 v_4$, and $W(0D; B) = v_1 + v_2 + v_3 + v_4 + q$.

Then

$$P_B(q, \{v_1, v_2, v_3, v_4\}) = W(2D; B) - q W(0D; B),$$

gives the exact critical manifold. (First obtained by Wu (1979)).
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J-S reformulated the cluster representation as a loop model, adapted to the lattice geometry.

Many details need sorting to build the TM. Different tricks typically needed for each lattice.

For the kagome lattice with $q = 1$ they can get to bases of size 7 in this way, giving the result quoted above.

Convergence is very fast. At least $O(1/|B|^4)$ often even faster than $O(1/|B|^6)$.

Another exact method for cases that can be exactly solved.

Fails to solve most cases that we’ve previously been unable to solve, but does provide lots of extra information (e.g. antiferromagnetic regime, Beraha number solution).

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Connection with integrability?
**Analyticity**

- **The Baxter approach**
  - Key parameter – spatial anisotropy. Y-B eqn. is satisfied by Boltzmann weights on the solution manifold.
  - Analyticity of local weights lift to thermodynamic quantities.
  - In the CFT approach, we have continuum critical scaling, and analyticity resides in the co-ordinates $z = x + iy$.
  - Correlation functions are holomorphic/anti-holomorphic functions of $z$, $\bar{z}$.
  - Recent developments link these.
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- Lattice model: identify discretely holomorphic observables whose correlators satisfy a discrete version of the C-R equations.
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Nienhuis’s $O(n)$ Loop Model.

- A gas of dilute non-intersecting loops.
- Key holomorphicity eqn. is a discretized contour integral.
- Let $G$ be a lattice.
- Let $F(z_{ij})$ be a c-v fn. defined on mid-points $z_{ij}$ edges $(ij)$.
- $F$ is discretely holomorphic on $G$ if

$$\sum_{(ij) \in \mathcal{F}} F(z_{ij})(z_j - z_i) = 0$$

where the sum is over the edges of each face $\mathcal{F}$ of $G$.
- For a square lattice it reduces to

$$F(z_{12}) + iF(z_{23}) + i^2F(z_{34}) + i^3F(z_{41}) = 0.$$
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A self-avoiding walk on the honeycomb lattice, starting and finishing on a mid-edge.

These are known to 105 steps (Iwan Jensen 2006)
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Figure: The figure shows the domain of width $T$ and height $2L$. Walks start at point $a$ and finish internally, or on the $\alpha$, $\beta$ or $\varepsilon$ ($\bar{\varepsilon}$) wall. Corresponding g.f.'s $A(x)$, $B(x)$, $E(x)$.
SMIRNOV’S HEXAGONAL LATTICE OBSERVABLE.

- The holomorphic observable is
  \[ F_z(x) = \sum_{\omega \in \Omega: a \rightarrow x} e^{-i\sigma W_\omega(a,x)} z^{l(\omega)}. \]
  
  - \( \omega \) is a walk from boundary point \( a \) to \( x \) in \( \Omega \). \( \sigma \in \mathbb{R} \) and \( z \geq 0 \).
  
  - \( l(w) \) is the \( |w| \), and \( W_\omega(a, b) \) is the rotation when \( \omega \) is traversed.
  
  - When \( z = z_c = 1/\sqrt{2 + \sqrt{2}} \) and \( \sigma = 5/8 \), \( F_{z_c} \) is discretely holomorphic, and satisfies
    \[ (p - v)F_{z_c}(p) + (q - v)F_{z_c}(q) + (r - v)F_{z_c}(r) = 0, \]
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**Smirnov’s Hexagonal Lattice Observable.**

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  where \( p, q, r \) are the mid-edges of the three edges adjacent to \( v \).
Recall \((p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0\).
Now sum this over all vertices in the domain.

- The inner mid-edges don’t contribute.
- The domain has a N-S symmetry
- The winding number of walks hitting the boundary is known

\[
\cos\left(\frac{3\pi}{8}\right) A_{T,L}(x_c) + B_{T,L}(x_c) + \cos\left(\frac{\pi}{4}\right) E_{T,L}(x_c) = 1.
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Figure: Bad picture with nice inset of $\cos\left(\frac{3\pi}{8}\right) A_T(z) + B_T(z)$ for honeycomb lattice walks in a strip of width 1, \cdots, 10.
There is no corresponding equation for SAW on other lattices.

For the square lattice, Cardy and Ikhlef found a similar observable. The model describes osculating SAW with asymmetric weights.

In the scaling limit, all SAW models should be identical, so “something similar" should be true for SAWs on other lattices.

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Figure: Square lattice \( \cos \left( \frac{3\pi}{8} \right) A_T(x) + B(x) \) for walks in a strip of width 1, \( \cdots, 15 \).
Conjecture (best estimates of $x_c$):

$$1 = c_A(T)A_T(x_c) + c_B(T)B_T(x_c),$$

Successive widths $(T, T + 1)$ give $c_A(T)$ and $c_B(T)$.
(Square lattice $T \leq 17$, triangular lattice $T \leq 11$).
Extrapolate:

$$\lim_{T \to \infty} \frac{c_A(T)}{c_B(T)} = \cos \left( \frac{3\pi}{8} \right)$$

to 6 sig. digits. Hence

$$\cos \left( \frac{3\pi}{8} \right) A_T(x_c) + B_T(x_c) = \text{const.} + \text{correction}$$

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To estimate $x_c$ we solve

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Successive values of $T$ give

$$x_c(T) = x_c(1 + O(1/T^{13/4})).$$

Extrapolate $x_c(T)$ and find

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(c.f. old conjecture of G. that $x_c$ is a root of $581x^4 + 7x^2 - 13 = 0$, giving $0.37905227775317290\ldots$),

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**Jacobsen’s Method for $O(n)$ Model**

- Semi-infinite cylinder of circumference $L$.
- Earlier work usually done on finite rectangles.
- Set up TM for SAWs, with weights $z^n$, ($n$ monomers).
- Compute leading eigenvalue of the TM in two different sectors:
  - (i) with an (open) strand from one end of the cylinder to the other. (A SAW with the ends at opposite ends of the cylinder).
  - (ii) with no propagating loop strands. Basically the SAP problem.
- The weight of a loop is taken to be $n = 0$, but loops winding around the cylinder get a different weight $n'$. 
- If one sets $n' = n$ this would just be the ground state sector. However one takes $n' = -\sqrt{2 - n}$ for a reason explained below.
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The f.e/site is \( f = -(1/L) \log(\Lambda_{\text{max}}) \).

\( f_0 \) is the ground state f.e., and \( f_i \) are the f.e’s in other sectors. From CI,

\[
f_i - f_0 = \frac{2\pi x_i}{L^2} + o(L^{-2}),
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where \( x_i \) is a critical exponent.

The exponent for paths in both sectors are known from CG arguments. The sector (2) exponent varies with \( n' \), which is chosen so that the exponents are equal.

Therefore one obtains, right at the infinite-size critical point

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f_2 - f_1 = o(L^{-2}).
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Define a finite-size critical point \( z_c(L) \) by finding the monomer fugacity s.t.

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• Define a finite-size critical point \( z_c(L) \) by finding the monomer fugacity s.t. \( f_2(L) = f_1(L) \) then the finite-size corrections to \( z_c(L) \) will be very small.
These corrections turn out to be exactly zero for solvable models (like that of Nienhuis on the hexagonal lattice), whereas for square and triangular SAWs they turn out to go like $1/L^4$ with subdominant $1/L^6$, $1/L^8$ etc terms.

So we systematically extrapolate to eliminate terms $O(1/L^6)$, $O(1/L^8)$, $O(1/L^{10})$, .... In this way the current result for the square lattice is $x_c = 0.3790522777533(2)$.

(From conjecture, $x_c = 0.37905227775317290....$)

One piece of evidence against the square SAW conjecture is that I’d expect the polynomial $581x^4 + 7x^2 - 13$ to be a factor when solving the TM equations, just as the corresponding result for the hexagonal lattice, $2x^4 - 4x^2 + 1$ is for the hexagonal lattice.
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This is a parallel development to our idea of adapting the Duminil-Copin/Smirnov identity that is exact on the hexagonal lattice to the square and triangular lattices.

In that case the relevant correction terms appear to decrease as $O(1/L^{k+1/4})$, $k = 2, 3, \ldots$, so convergence is not as rapid.
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In that case the relevant correction terms appear to decrease as $O\left(\frac{1}{L^{k+1/4}}\right)$, $k = 2, 3, \ldots$, so convergence is not as rapid.
Jesper’s final comment: ‘It took me a very long time to get there, basically because this set of ideas all started on finite rectangles, and I was slow to realise that it was more effective to turn it into an eigenvalue problem on a semi-infinite cylinder.”
**Conclusion**

- Four methods, all exact for some situations, not for others. Why?
  - Non-D-finiteness is one answer.
  - Maybe natural boundaries is another answer?
  - In any event, we now have a powerful suite of tools to obtain increasingly precise numerical estimates of critical parameters, and equally significantly, to give insight into the solvability of the underlying problem.
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