Towards the Microscopics of Condensates

A. Gorsky (Institute for Information Transmission Problems, Moscow)

May 8, SCGP

Based on works, Bulycheva, Nechaev, A.G. 1409..
Milekhin, A.G. 1412..., Milekhin, Sopenko, A.G. To appear,
Avetisov, Nechaev, Valba, A.G. To appear
Plan of the talk

1. Introduction. Where the question comes from?

2. Squark condensate in 5D SUSY QED and QCD from the torus knot invariants

3. Critical behavior in the «holomorphic» topological ensembles and knots

4. Critical behaviour in the « instanton-antiinstanton» ensembles via colored network

5. Conclusion
Examples of condensates

- Gluon condensate in QCD $<\text{Tr}G^2>$ (SVZ,79)
- Chiral condensate in QCD at small density and Cooper pair quark condensate at large density.
- Potential of the Higgs field in the Standard Model is highly sensitive to the ratio of t-quark mass and Higgs boson mass. Hint for the composite Higgs $H=\bar{\text{t}}\text{t}$?
- Not many exact results (Casher-BAnks). Somehow appear in the instanton-antiinstanton ensemble.
Examples of condensates. SUSY

- Holomorphy - «instantons» contribute the condensate
- Longstanding puzzle concerning the gluino condensate in N=1 SYM. If one compact dimension- fractional instantons contribute
- Squark and gluino condensates are related via Konishi anomalies
- Exact result. Squark and monopole condensates vanishes at Argyres-Douglas point in N=1 SQCD.
New tools

- New invariants of knots. Khovanov homologies and superpolynomials which generalize Jones and HOMFLY polynomials
- Seberg-Witten solution to N=2 SYM. Nekrasov partition sums. Explicit results for the instantons sums in the Omega-background
- Topological phases of matter. Classification via ground state degeneracy+ holonomy of Berry phase or via entanglement entropy
Knot invariants

- \( J(q,K) = \langle W(K) \rangle \) in SU(2) 3d Chern-Simons theory — Jones polynomial of knot K (Witten, 89). Can be generalized to all SU(N) groups-
- HOMFLY polynomial \( H(a,q,K) \), \( a=q^N \)
- Generalization to superpolynomial \( P(a,q,t,K) \) (Dunfield, Gukov, Rasmussen 04)
- All knot polynomials are particular indexes
  \[ P(a,q,t,K) = \dim H_{\{ijk\}} a^i q^j t^k \] and count the multiplicities of the BPS states (Gukov, Schwartz, Vafa 04)
Torus knots can be drawn on the torus surface. $T(n,m)$ corresponds to two windings around cycles.
Knot invariants. New approaches

- The old evaluation - it is the vev of electric Wilson loop along the knot in 3d CS. Is there the S-dual «magnetic» version of the knot invariants? (Witten 10, Witten, Gaiotto 11). Approach — knot invariants count instantons in 4d and 5d gauge N=4 SUSY gauge theory.

- The superpolynomials of the torus knots are expressed as very specific integrals over moduli space of points in C^2 (E.Gorsky-Negut , 13). Way to instantons.
5D SQED and SQCD

- Consider the U(1) 5d SUSY gauge theory with \( N_f=2 \) or \( N_f=3 \). One dimension is compact \( S^1 \). Add 5d CS term \( k \) AFF. Introduce Omega-deformation= two independent rotations (angular velocities) in \( R^4 \).

- There is the explicit answer for the instanton partition function in this theory due to Nekrasov localization

- Surprise. The condensate of the massless flavor « is sum over the invariants of the T(n,m) torus knots»
Some facts on 5d SQED

- The BPS particles in the theory are W-bosons, instantons. Due to the CS term the instanton charge induces the electric charge.
- Complicated dyonic instantons (both charges). Even more complicated states with 3 charges (+flavor). Not fully classified. Monopoles are loops (monopole particles lifted tp 5d).
- One-loop effect of all BPS particles in 5d D with compact dimension reproduces all instanton partition sum in D=4 SYM theory (Nekrasov-Lawrence).
Instanton-torus knot duality,

\[ \frac{e^{\beta M}}{(1 + a)\beta^2} \frac{d^2 Z_{nek}(q, t, m, M, m_a, Q)}{dM \, dm} \bigg|_{m \to 0, M \to \infty} = \sum_n Q^n (tq)^{n/2} P_{n,nk+1}(q, t, a) \]  

where \( m_a, m, M \) are masses of three hypemultiplets and \( Q \) is the counting parameter for the instantons. The mapping between the parameters at the lhs and rhs goes as follows

\[ t = \exp(-\beta \epsilon_1) \]  
\[ q = \exp(-\beta \epsilon_2) \]  
\[ a = -\exp(\beta m) \]  
\[ Q = \exp(-\beta / g^2) \]

Milekhin, A.G. 1412 and to appear
Instanton-torus knot duality

The superpolynomial is the complicated product in terms of the Young tableau.

\[ P(A, q, t)_{nk+1,n} = \sum_{\lambda:|\lambda|=n} \frac{t^{(k+1)} \sum q^{(k+1)} \sum a(1-t)(1-q) \prod^{0,0}(1-Aq^{-a't-1'}) \prod^{0,0}(1-q^{a't'})(\sum q^{a't'})}{\prod(q^a-t^{l+1}) \prod(t^l-q^{a+1})} \]

In this formula there is only one independent index — n (instanton number)
New findings

- The information about the knots is encoded in the condensate. Torus knots $T(m,n)$ are important. The physical identifications of the numbers: $n$- instanton charge, $m$ -electric charge

- The physical variable is expressed in terms of the sum over the knots. The first example of such situation!

- The composite defect seems to be relevant. $N$-instantons sitting at the top of each other, attached strings, and surrounding domain wall.
Interpretation

- The knot invariants describe the multiplicity of states at fixed 4d quantum numbers \((n,m)\).
- In some sense they count the 2d instantons on the nonabelian strings at fixed 4d instanton number. «Knotting the fermionic zero modes?»
- Dyonic instantons enter the game.
- Linking versus knotting?
HOMFLY for generic \((n,m)\) knots

- Consider the \(N_f=2\) theory with Lagrangian brane. Count the contribution of states with \((n,m)\) quantum numbers into condensate.

- Consider the \(SU(2)\) theory with \(N_f=4\). Two masses fixed, one mass vanishes, one is arbitrary. Expand the condensate in series in two quantum numbers.

- Consider \(N_f=2\) \(U(1)\) with fractional 5d CS number \(k=m/n\). Extract \(n\)-instanton contribution.
Complimentary views on the instanton sums
Bulycheva, Nechaev, A.G. 1409

Nekrasov partition function as a path integral

Equivariant integrals over $n$-instanton moduli space for 5d $U(1)$ SUSY QED

Summation over Young diagrams

$q$-deformed (magnetic) random walks

Weighted partition functions of single 1D Dyck paths

Extremal statistics of 1D "vicious" random walks

Superpolynomials of $T_{n,n+1}$ torus knots
Critical behavior in the instanton ensemble (the use of random walk picture)

The surface of the phase transition

\[ \left(1 + Q - Qe^{m_\alpha \beta}\right)^2 = 4Q \]

A lot in common with critical behavior at Argyres-Douglas point in D=4 SQCD (Vainshtein, Yung, A.G. 02)
The critical behaviour is extracted from the explicit expression for the condensate in self-dual case

$$F(q, a, s) = \frac{A_q(s; s(1-a))}{A_q(s/q; s(1-a)/q)}$$  \hspace{1cm} (13)$$

where $A_q(s; s(1-a))$ is the extension of the $q$-Airy function $A_q(s)$ defined in (8). The function $A_q(s; s(1-a))$ reads

$$A_q(s; s(1-a)) = \sum_{k=0}^{\infty} \frac{q^{k^2} (-s)^k}{(q; q)_k (-s(1-a); q)_k}$$  \hspace{1cm} (14)$$

$q$-- angular velocity, $s$-- gauge coupling, $a$-- mass
Figure 5: Relief of the generating function $F(a, s)$ above the $(s, a)$–plane for different fixed values of the area fugacity, $q$. For $q = 1$ the phase 1 corresponds to short paths, and the phase 2 – to long and rather wrinkled paths. For $q \neq 1$ one sees in the $(s, a)$–plane the cascade of transitions with the Airy-type asymptotics.
Physics behind the phase transitions. Different Interpretations

--- Partial unknotting. Different torus knots dominate

--- Different values of the instanton charges dominate

--Different trajectories in the random walks dominate
Motivation. Topological picture behind the QCD-like condensats in the Instanton-antiinstanton ensemble.

There is the matrix model, modelling the spectrum of the Dirac operator in the $I-I$ ensemble. The condensate can be obtained via Casher-Banks relation or equivalently it is proportional to the $I-I$ connections in the matrix model of Verbaarschot et.al
Hamiltonian of the network with two colors

\[ H = c(N_{\text{black triangles}} + N_{\text{white triangles}}) + \]

constraint - valence conservation at each vertex,
mimics the conserved number of zero modes

\[ Z_2 \] symmetric Hamiltonian

It belongs to the Levin-Wen type Hamiltonians describing the topological order. Two type of contraints - «magnetic» and «electric» (faces and vertices)
Surprising results of the numerical simulations.

1. For any value of $c (=2d$ cosmological constant) there is separation of the colors in two clusters. In the language of 2d gravity — magnetized baby universe

2. There is the «conformal plataeu» when the number of Black-White connections (analogue of chiral condensate) does not depend on $c$

3. There is the critical valence when the plataeu appears $V_{\text{crit}} = 20$. It corresponds to the critical dimension of the target space $D_{\text{crit}} = 10$
Attempt of explanation and lessons for QCD

- Separation of colors — effect of «axial anomaly» in 2d gravity. There is similar unexplained separation of chiralities into «3d sheets» in lattice QCD

- Plataeu — entropic phenomena. Long-range entanglement. Some analogue of density independence of chiral condensate in QCD

- Critical dimension $D=10$ for the plateau. Holography via superstring? $D=10$ is the critical dimension for superstring
Conclusion

• Just touch the tip of the iseberg. Some surprises

• The «knotting» between electric degrees of freedom and instantons is important for the condensate formation

• Topological order behind the vacuum in «instanton -antinstanton» ensemble?

• A lot of open questions............