Homotopy of $L_\infty$ homom.

We imitate Sullivan’s app. in CDGA case but in $L_\infty$ case

step is actually simpler.

$$\hat{C} = \prod_{t \in R_0} IR[y]$$

More case,

$R_0$: moduli of closed Reeb orbit

**Def.**

$$C^\infty([0,1],\hat{C})$$

$$= \prod_{t \in R_0} C^\infty([0,1],IR[y]) \oplus \prod_{t \in R_0} C^\infty([0,1],IR) d\phi \circ C[y]$$
Its element is a formal infinite sum

\[ \sum_{y} f_{y}(t) \varnothing y \] + \[ \sum_{y} g_{y}(t) dt \gamma \varnothing y \] = \star

for any co-function \( \varnothing y \), \( y \in \mathbb{C} \)

\[ M_{i}(x) = \sum \frac{\partial f_{i}}{\partial t} \varnothing x \] + \[ \sum f_{i}(x) M_{i} \gamma \varnothing y \] - \[ \sum g_{y}(x) dt \alpha \varnothing y \]
\[ H_{\hat{i}} = \sum f_{y_i}^1 (x) + \sum g_{\theta_i}^j (x) \]

\[ M_k(x_1, \ldots, x_n) \]

\[ = \sum_{x_i} f_{y_i}^1 - f_{y_i}^2 M_k(x_{i+1}, \ldots, x_n) \]

\[ + \sum_{i=1}^k \sum_{0 \leq i < k} (-1)^i f_{y_i}^1 - f_{y_{i+1}}^i g_{\theta_i}^j f_{y_{i+1}}^i - f_{y_i}^j \]

this is \( M_k \) of \( \hat{C} \)

\( (H)^4 \) is a kanze sign

\[ + \sim 1 + \deg x_i + \ldots + \deg x_{i-1} \]
Lemma 1

$C^0([0,1], \mathbb{C})$ is an $\mathbb{R}$-alg.

 :) Easy

Def

$Ev_t : C^0([0,1], \mathbb{C}) \rightarrow \mathbb{C}$

$Ev_t : c([0,1]) \rightarrow c$

$f \in c([0,1]) \rightarrow f(t)(x)$

$f \in c([0,1]) \rightarrow 0$

Len: $Ev_t$ defines a linear $\mathbb{R}$-homeo
Def: \( \mathcal{C}_1 \rightarrow \mathcal{C}_2 \) is a homomorphism.

\( f \sim g \) (homotopic)

\((=) \exists \ H : \mathcal{C}_1 \rightarrow C^0([u_i, v_i] \mathcal{C}_2) \)

\( L^0 \) homm.

\( \text{Ev}_0 \circ H = f, \)

\( \text{Ev}_1 \circ H = g \).
Lemmas

1. Homotopy is an equivalence relation.

2. \( f \sim g \implies \begin{cases} h \circ f \sim h \circ g \\ f \cdot h \sim g \cdot h \end{cases} \)

This is proved (in its A5 analogue) in Fouul's book on LAG. From their Chapter 4.

(The proof in the case is similar.)

Only a few words, on \( f_1 \sim f_2 \), \( f_2 \sim f_3 \)

\[ \implies f_1 \sim f_3 \]
\[ f_i : \hat{C} \rightarrow \hat{C} \quad \text{for} \quad i = 1, 2, 3 \]

\[ \mathcal{C}(\hat{C}, \hat{C}') \bigvee \mathcal{C}^\omega(\hat{C}, \hat{C}') \]

\[ \lor \]

\[ (\exists f_i \in \mathcal{C}(\hat{C}')) \land \exists \delta \in \mathcal{C}(\hat{C} ') \]

\[ \neq \]

\[ (\exists f_i \in \mathcal{C}(\hat{C}')) \neq (\exists f_i \in \mathcal{C}(\hat{C}')) \]

\[ f_i \neq f_i \]

\[ f_0 (1) = f_i (0) \]

\[ f_1 \sim f_2, \quad f_2 \sim f_3 \]

\[ H_1 : \hat{C} \rightarrow \mathcal{C}(\hat{C}, \hat{C}') \]

\[ H : H_1, H_2 \quad \text{define} \]

\[ H = H_1 \vee H_2 : \hat{C} \rightarrow \mathcal{C}^\omega(\hat{C}, \hat{C}') \bigvee \mathcal{C}^\omega(\hat{C}, \hat{C}') \]
that is an $L^2$-hom

\[ \mathcal{E}' \xleftarrow{\text{Ev}_2} C^0(\mathcal{E}, \mathcal{E}') \xrightarrow{\text{Ev}_1} \mathcal{E}' \]

\[ \mathcal{E}' \xleftarrow{C^0(\mathcal{E}, \mathcal{E}')} \mathcal{E} \]

If it exists we are done.
Construction of \( \mathcal{C} \) is by induction on energy, similar to the proof of

\[
\text{The } f : \hat{\mathcal{C}} \rightarrow \hat{\mathcal{C}}', \text{ hence } f_! : \hat{\mathcal{C}} \rightarrow \hat{\mathcal{C}}'
\]

induce isomorphism in \( M \) homology.

- \( \exists \phi : \hat{\mathcal{C}}' \rightarrow \hat{\mathcal{C}} \text{ is hom.} \)

so \( \phi \circ \phi \text{ and } \phi \circ \phi \text{ is id} \)

This is an analog by of Whitehead theorem in homotopy theory.
From geometry to algebra.

\((N,\overline{S})\) contact manifolds,

\[ \Theta, \Theta' \text{ forms } \overline{S} = \ker \Theta = \ker \Theta', \quad \Theta \wedge \Theta' \text{ never } 0 \]
\[ \Theta' \wedge \Theta' \text{ never } 0 \]
\(\dim N = 2n+1\)

\(\Theta' = \lambda \Theta\) many cases \(k \geq 1\).

\[ h : \mathbb{R} \times N \to \mathbb{R} \]
\[ h = 1 - 0 \ll 0 \]
\[ h = 0 \gg 0 \]

\[ \frac{\partial \Theta'}{\partial \Theta} \geq 0. \]

\[ w = d(e^{\varphi} \Theta) \quad \text{sym form a } \mathbb{R} \times N. \]
\( J : \text{almost complex str. on } \mathbb{R}^2 \times \mathbb{N} \)
compatible with \( \omega \).

\( \delta \circ \omega \cdot J \) 
\( \rightarrow \) 
\( J_1 \) 
\( \text{IR inv. almost cp. str. on } \mathbb{R}^2 \times \mathbb{N} \)
compatible with \( \text{d} (e^\omega \Theta) \)

\( \delta \circ \omega \cdot J_2 \) 
\( \rightarrow \) 
\( J_2 \) 
\( \text{IR inv. almost cp. str. on } \mathbb{R}^2 \times \mathbb{N} \)
compatible with \( \text{d} (e^\omega \Theta) \)

Use \( \mathcal{U} \) 
\( \rightarrow \) 
\( \mathbb{R} \times \mathbb{N} \) 
\( J \) holomorphic

\( H \) is \( \langle f_3 (x_1, x_2, x_3) \rangle \)
to obtain $f_h: \hat{C}(\theta) \hat{\otimes} \hat{C}(\theta') \rightarrow \hat{C}(\theta')$

Let hom $C(\theta); \mu_{h, \theta} \rightarrow (C(\theta'), \mu_{h', \theta})$

Take another $h, j$, (and another perturbation) we obtain $g_h: \hat{C}(\theta) \hat{\otimes} \hat{C}(\theta') \rightarrow \hat{C}(\theta')$

This $f=(f_h)$ is homotopic $h$

$g=(g_h)$ as $L_5$ homomorphisms
First let me explain how people usually prove

\[ f : \hat{E}_C(0) \longrightarrow \hat{E}_C(6') \]

is \underline{chain homotopic} to

\[ g : \hat{E}_C(0) \longrightarrow \hat{E}_C(6). \]

\[ h, j \longrightarrow f \]

\[ h', j' \longrightarrow g. \]
$H : [0,1] \times \mathbb{R}_0 \times N \to \mathbb{R}$

\[
\begin{cases}
    H(0, \sigma, x) = \tilde{h}(\sigma, x) \\
    H(1, \sigma, x) = \tilde{h}'(\sigma, x) \\
    H(\delta, \sigma, x) = 1 & \sigma < 0 \\
    H(\delta, \sigma, x) = h & \sigma > 0 \\
    \frac{\partial H}{\partial \sigma} \geq 0
\end{cases}
\]

$\mathcal{F} = \{ \tilde{H} \}$ [0,1] parameterized family of almost complex structures
$J_5$ is compatible with $d(e^{\theta} H(s, \cdot \cdot \cdot) \Theta) = \omega_5$

$J_0 = J, \quad J_1 = J'$

$J_5 = J_1, \quad \sigma < 0$
$= J_2, \quad \sigma > 0$

(If $J_1, J_2$ used to define $\hat{C}(0)$
$\hat{C}(\Theta)$.)

$M_0(x_1, \cdots, x_5; J_5)$ moduli of $J_5$ holomorphic maps
We consider

\[
\bigcup_{\mathcal{S}} \times M_0(\mathbf{y}_1 - \mathbf{y}_0; \mathbf{y}' : \mathcal{S})
\]

\[
= M_S(\mathbf{y}_1 - \mathbf{y}_0; \mathbf{y}' : \mathcal{F})
\]

\[
d \in M_0(\mathbf{y}_1 - \mathbf{y}_0; \mathbf{y}' : \mathcal{F}) = 1 + M_0(\mathbf{y}_1 - \mathbf{y}_0; \mathbf{y}' : \mathcal{I})
\]

We consider the case when LHS = 0

We put

\[
F_h(\mathbf{y}_1 - \mathbf{y}_0) = \sum_{\mathbf{y}'} \# M(\mathbf{y}_1 - \mathbf{y}_0; \mathbf{y}' : \mathcal{F}) (\mathbf{y}' /)
\]
The first guess.

Use $F^2$ at $F$, $S'$ to obtain a chain homotopy between $F$ and $G$.

We can do it by studying the boundary of $M(x, x', Y; \partial)$ in case it is 1-dimensional.

Unfortunately, the guess is not correct.
Let us study $J M (R_i - 0, R'_i, J')$ in case it is 1-dimensional.

There are 3 types.

Type 0

\[ M (R_i - R'_i, R'_i, J'_i) \rightarrow f \]

\[ M (R_i - R'_i, R'_i, J'_i) \rightarrow q \]
Type 0 \rightarrow \theta - f \text{ this is OK}

Type B \rightarrow \forall \alpha . \exists \beta . \sigma . \text{ something we construct from } \phi.

\text{this is also OK}

\text{The problem is } \exists \phi \text{ of type A}.
This part gives \( \mu_3(x_1, x_2, x_3) \). \( \delta_x \)

\[ c = 1 \]

This is not \( \langle F_2(x_1, x_2), x_1 \rangle < F_1(x_3), x_2 \rangle < F_3(x_4, x_5, x_6), x_3 \rangle \)
This part is a count of

\[ \bigcup S \text{ is } M(\gamma_1, \delta_2; \gamma_1', T_5) \times M(\gamma_2, \delta_2'; \gamma_1', T_5) \]

\[ \times M(\gamma_4, \delta_5, \delta_4; \delta_3', T_5) \]

Among

\[ M(\gamma_1, \gamma_2; \gamma_1', T_5), M(\gamma_2, \gamma_2'; T_5) \]

\[ M(\gamma_4, \delta_5, \delta_4; \gamma_3', T_5) \]

two of them has dim 0, one is dim -1

So this count is something like

\[ \langle f_2^*(\delta_2, \delta_2'), \gamma_1' \rangle \langle f_3^*(\gamma_3), \gamma_2' \rangle \langle F_3(\gamma_4, \delta_5, \delta_4), \delta_3' \rangle \]
Note \( F_3 \) is defined by \( M(x, y; z; g) \)

\( f_2 \) is defined by \( M(x, y, y'; z) \) for a \( f \) fixed.

5. The curve is not defined \( F_{-i} \) of \( f = f_0 \)

and \( g_+ = f_+ \). So in place we define

\[ \langle \mathcal{A}, (x_{i_0}, ..., x_{i_k}) \mid (y_1, ..., y_k) \rangle \]

by
\[ k \cup \{ s \} \times M(\gamma_1, \gamma_i, \gamma_s) \rightarrow M(\gamma_{2k}, \gamma_{1k}, \gamma_s) \]

\[ s \in \{0,1,2\} \]

\[ \gamma_1, \gamma_2, \gamma_3 \]

\[ \gamma_{1k}, \gamma_{2k}, \gamma_{3k} \]

\[ N \times N \rightarrow M \times M \text{ with disconnected } \vec{12} \]
We define

\[ M(\gamma^1, \ldots, \gamma^n; \mathcal{D}) \]

as

\[ \bigcup_{S \in e(\omega_1)} \bigtimes_{i} M(\gamma^i; \mathcal{D}_i) \times \cdots \times M(\gamma^n; \mathcal{D}_n) \]

and

\[ F(\gamma^i - \gamma^j) = \sum \sum_{I_i \in I \cap \Omega} \bigtimes_{l \in I_i} \bigtimes_{j \in I_i} \bigtimes_{k \in i, \theta} \bigtimes_{\alpha \in \mathcal{D}_i} \bigtimes_{\gamma \in \mathcal{D}_j} \theta \]

where

\[ I_i \cap I_j = \emptyset \quad \cup I_i = \{1, \ldots, m\} \]

\[ F: \prod_{i=1}^{m} \mathcal{E} \to \prod_{i=1}^{k} \mathcal{E} \]
We can show
\[ A^\# \cap H^0 = 0 \rightarrow f \] (i.e. $f$ is chain homotopic to $g$)

if everything is torsionless.

This is a "picture proof" of the claim that
\[ f : EC(\alpha) \rightarrow EC(\beta) \]
\[ g : EC(\alpha) \rightarrow EC(\beta) \]
are cochain homotopic.
There are two points one need to clarify.

1. Can one say $A \sim_0 B$ is homotopic as $L$-algebra? This point does not seem to imply it. (Need to study rel. between $L$ and coalgebra structure.)

2. It is very difficult to achieve transversality this way. (Problem of self-glueing)
Difficultly

Consider the case \( \log Y = \log Y' - 1 \) for \( r \in R_6 \)

\[ \dim M(r, r'; J) = -1 \]
\[ \dim M(r, r'; \theta) = 0 \]

We consider the following configuration.
$u_2, u_3 \in M(\mathbb{R}, \mathbb{R}'; T_5)$ for some $s$

$u_1$ is branched double cover of trivial cylinder

$\mathbb{P}^1 \to \mathbb{P}^1$

$\mathbb{Z} \to \mathbb{Z} \oplus (\mathbb{Z} \cdot (1 - t))$

$\infty$ is double branch point
This configuration appears in

\[ M(\mathbb{R}; \mathbb{R}^2; \mathbb{E}) \] and is identified with

\[ \bigvee \mathbb{N} \times M(\mathbb{R}; \mathbb{R}^2; J_5) \rightarrow M(\mathbb{R}; \mathbb{R}^2; J_5) \]
Note \( \dim M((x,x', T_5)) = -1 \)

So

\[
\dim \left( \bigcup_{S} U(S) \times M((x,x', T_5)) \times M(x,x', T_5) \right) = 1 + -1 + -1 = -1
\]

But this is non-empty \( M((x,x', 2)) \)

\[
= \bigcup_{S} U(S) \times M((x,x', T_5)) \quad \text{is non-empty}
\]

\[
\dim M((x,x', 2)) = 0
\]
This problem is called a problem of self-growth and is known to SF7 people at least at this stage.

Maybe first observed by Hutchings during the study of Novikov homology (finite-dimensional case).

A similar problem appears in lag. Then theory. See TWO 2009 book Subsection 7.2.1.
In that case the problem is solved in Tero 2009 book, Section 7.2. That method can be used in our situation.

See arXiv:1704.01848v1, Subsection 15.7.4.

Here I will explain an idea to resolve it. (To work it out we need to develop certain language and so it is postponed to later.)
We consider

\[ M(\chi_1, \ldots, \chi_n : \mathcal{D}) \arrow [0, 1] \]

\[ (S, (\mathcal{Z}, \mathcal{W}, \eta)) \arrow [S] \]

It is hot.

The problem is that the fiber product

\[ M(\chi, \chi : \mathcal{D}) \times M(\chi, \chi : \mathcal{D}) \]

is not transversal. So \( p^\Lambda \subseteq [0, 1] \).
Can never be transverse to itself

We use de Rham model for cohomology of $\mathbb{C}^n$ rather than singular homology.

Why? Wedge product of differential forms is always well defined but intersection pairing of chains has transversality problem.
\[ M(\gamma, \gamma'; z) \rightarrow [0, 17] \]

\[ \text{Fundamental chain in de-Rham model} \]

\[ \delta(s - s_0) \]

\[ \text{delta function} \]

Problem: in this form, it is not smooth but is a distribution. \[ \delta(s - s_0) \delta(s - s_0) \] is not defined (product of distributions).
Iden to smooth delta function

\[ \delta_{\lambda}(s) = \int_{-\infty}^{\infty} \frac{e^{-\pi s^2 x^2}}{x} \, dx \]

How we can do it?
Continuous family of perturbations

To perturb $M(x, y; \delta)$ we modify $x = \|x - N\|

$\overline{\delta u} = 0 \quad \overline{\delta u} = s(u)$

$s(u) \subset C^0(\bar{\Omega}, \Lambda^0 \otimes \mathbf{U}^* T x) \quad \mathbf{U}^* T x$

We introduce auxiliary space $B$ (finite dimensional space)

and $s(u, \beta)$ is dependent family of perturbation.

$C^0(\bar{\Omega}, \Lambda^0 \otimes \mathbf{U}^* T x) \quad \beta < \beta_3$
\[ M(\mathfrak{r}, \mathfrak{r}', \mathfrak{d}; \beta) = \left\{ (\mathcal{S}, \mathcal{P}, (\mathfrak{z}, \mathfrak{w}, \mathfrak{n})) \mid \begin{array}{l}
\mathcal{S} \in \mathcal{C}(\mathcal{U}, \mathcal{I}) \\
\mathcal{P} + \beta \\
\mathfrak{u} : \mathfrak{z} \times \mathfrak{w} \to \mathfrak{M} \times \mathfrak{N} \\
\delta \mathfrak{U} = \mathcal{S}(\mathfrak{U}, \mathfrak{P}) \\
asymptotic \ and \ at \ \mathfrak{w}
\end{array} \right\} \]

\[ \dim M(\mathfrak{r}, \mathfrak{r}', \mathfrak{d}; \beta) = \dim M(\mathfrak{r}, \mathfrak{r}', \mathfrak{d}; \beta) + \dim \beta \]

\[ \beta : M(\mathfrak{r}, \mathfrak{r}', \mathfrak{d}; \beta) \to \mathcal{C}(\mathcal{U}, \mathcal{I}) \]

\[ \Psi : (\mathfrak{p}, (\mathfrak{z}, \mathfrak{w}, \mathfrak{n})) \to \mathfrak{S} \]
By choosing $d$ in $\mathcal{P}$ large we may assume

$M(r, y; \mathcal{J}; \beta)$ is smooth (of A) and $M(r, y; \mathcal{J}; \beta) \xrightarrow{\text{ev}_s} \mathcal{C}_0.17$ is a submersion.

Choose $X$ a differential form of top degree on $\mathcal{P}$ and of compact support satisfying $\mathcal{S}X = 1$.

$\mathcal{P} \xrightarrow{\text{ev}_p} M(r, y; \mathcal{J}; \beta) \xrightarrow{\text{ev}_s} \mathcal{C}_0.17$

$X \xrightarrow{\text{pull back}} \text{ev}_p^*X \xrightarrow{\text{push out}} \text{ev}_s^*! \text{ev}_p^*X$
This is a smooth differential form

\[ \text{dim } N(\partial, Y' : J) = 0 \Rightarrow ev_s! ev_I^* X \triangleq 0 \text{ form} \]

\[ \text{dim } N(\partial, Y : J) = -1 \Rightarrow tv_s! ev_I^* X \triangleq 1 \text{ form} \]

Now for \( M(\gamma, \gamma_1, \gamma_2) \), we choose such \( \beta, X \) in a consistent way and put

\[ (4) \]
given a \( \mathbb{L}^\phi \)-hom \[ H_\phi (\gamma_i - \gamma_t) = \sum \varepsilon \varepsilon \nu X (\gamma) \]

when \( \dim M (\gamma_i - \gamma_t; \gamma') = 0 \) or \( 1 \).

\[ \beta - M (\gamma_i - \gamma_t; \beta) \rightarrow \mathbb{C}_u N \]

This is an element of \( \mathbb{C}^\phi (\mathbb{C}_u N, \mathbb{C} \mathcal{O}_0) \)

This gives a \( \mathbb{L}^\phi \)-hom \[ \mathcal{H}_\phi : \mathcal{C}(\theta) \rightarrow \mathbb{C}^\phi (\mathbb{C}_u N, \mathcal{C} \mathcal{O}_0) \]

This is the homotopy of \( \mathbb{L}^\phi \)-hom.
To make it work we need
story of continuous form of perturbation

It is written in
Kuranishi Structures and Virtual Fundamental
chains (FOOO)

The method used by Ishikawa in
arXiv: 1807.01419 is similar I think.