Supersymmetric Cardy formulae

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The following is (mostly) concerned with the “old” 4d/6d SUSY Cardy formula (“real fugacities”), first conjectured by [Di Pietro-Komargodski]. The techniques outlined here are more general however, and can be applied to the “new limit” [Choi-Kim-Kim-Nahmgoong; Cabo-Bizet-Cassini-Martelli-Murthy; · · · ], which seems to be connected to black hole microstate counting. Time-permitting, I will say a few words about this at the end.
Let the fun begin:

1. Motivation: 2d Cardy formula
2. Higher-dimensional Cardy formulae
   - Preliminaries
   - SUSY Cardy formulae: A conjecture
   - A (new) check from localization
3. Proof:
   - Overview
   - Effective actions
   - 5d Chern-Simons invariants
   - Chern-Simons couplings from effective action
   - Matching to perturbative anomalies on the Higgs branch
   - Examples
4. Global anomalies from effective action
5. Some Remarks
6. Summary and outlook
Motivation: 2d Cardy formula ($\rightarrow$ see earlier talk)

- Torus partition function:
  $$Z_{S_1 \times S_1}(\tau) = \text{Tr}_{\mathcal{H}_{S_1}} \left( q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \right), \quad \tau = \frac{i\beta}{2\pi}$$

- Cardy limit $\beta \rightarrow 0$ [Cardy]:
  $$\log Z_{S_1 \times S_1} = \frac{\pi^2 c}{3\beta} + O(\beta^0, \log \beta)$$

- Key Ingredient:
  - Modular invariance: $Z(\tau) = Z(-1/\tau)$

- Consequences/Features:
  1. Operator spectrum $\rho(\Delta)$ in the high-energy asymptotic region
  2. **Universality:** only dependent on $c$
  3. Under the AdS/CFT: maps to the universality of the Bekenstein-Hawking entropy of BTZ black holes [Strominger-Vafa]

- Recent resurgence (more details in e.g. earlier bootstrap talk)
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Summary and outlook
Preliminaries: Higher-dimensional Cardy formulae?

Some complications:

I. **Spacetime?** Natural choices: \(\mathbb{T}^d\), \(\mathbb{T}^2 \times S^{d-2}\), \(S^1 \times S^{d-1}\), \(S^1 \times \mathcal{M}_{d-1}\), ...

II. **Modularity?** Unknown/complicated in general (e.g. some recent progress for \(S^1 \times S^3\) in [Dedushenko-MF])

III. **Which anomalies?** Many anomaly coefficients.

IV. **Universality?** Dependence on spacetime/couplings seems complicated/non-universal

Does high-temperature universality exist in \(d > 2\)?
THIS LOOKS LIKE A JOB FOR SUPERSYMMETRY
Preliminaries: Higher-dimensional Cardy formulae?

SUSY:

- SUSY restricts possible background couplings (SUSY completion) $\leadsto$ essentially metric $\oplus$ SUSY partners.
- SUSY partition functions $\leftrightarrow$ geometric invariants
  
  \[ \text{[Closset-Dumitrescu-Festuccia-Komargodski]} \text{ (in 5d, [Imamura-Matsuno; Alday-Benetti-MF-Richmond-Sparks; \ldots])} \]

- Counterterm ambiguities under control $\leadsto$ unambiguous answer
  
  \[ \text{[Closset-Dumitrescu-Festuccia-Komargodski; Chang-MF-Lin-Wang]} \text{ (as opposed to the } \beta \to \infty \text{ limit [Closset-Di Pietro-Kim])} \]

- “Fewer” anomalies: Weyl $\leftrightarrow$ ’t Hooft

Proposals:

- “Spacetime”: $\mathcal{Z}_{S^1_\beta \times S^{d-1}} \equiv$ superconformal index

- Proposals by [Di Pietro-Komargodski] for 4d and 6d theories

- Recently: alternative, “complex high-temperature” limit
  
  \[ \text{[Choi-Kim-Kim-Nahmgoong; Cabo-Bizet-Cassini-Martelli-Murthy; \ldots]} \leadsto \text{ see later!} \]
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The 4d case

- $\mathcal{N} = 1$ SUSY index (turn off flavor fugacities for simplicity in 4d):

$$Z_{S^1_\beta \times S^3} \equiv Z_{S^1_\beta \times S^3(\omega_1, \omega_2)} \propto \text{Tr } \mathcal{H} \left[ (-1)^F e^{-L\{Q, Q^\dagger\}} e^{-\beta \sum_{i=1}^2 \omega_i(j_i + R)} \right]$$

- In the Cardy limit $\beta \to 0$, this SUSY partition function has the expansion [Di Pietro-Komargodski]:

$$\log Z_{S^1_\beta \times S^3} = \frac{\pi^2}{6\beta} \frac{\omega_1 + \omega_2}{\omega_1 \omega_2} \kappa + O(\beta^0, \log \beta)$$

- $\kappa$ related to the anomaly coefficient $k$ for the mixed gravitational-R-symmetry by

$$\kappa = -k$$

- The anomaly coefficient $k$ appears in the anomaly polynomial 6-form as

$$l_6 \ni \frac{k}{48(2\pi)^3} F_R \wedge \text{tr} (R \wedge R)$$

- By SUSY, $\kappa$ and $k$ are in turn related to the 4d conformal anomalies as

$$\kappa = -k = 16(c - a)$$
A conjecture in 6d

- \( \mathcal{N} = (1, 0) \) 6d index:
  \[
  \mathcal{Z}_{S_1^6 \times S_5} \equiv \mathcal{Z}_{S_1^6 \times S_5(\omega_1, \omega_2, \omega_3)} \propto \text{Tr} \, \mathcal{H} \left[ (-1)^F e^{-L\{Q, Q^\dagger\}} - \beta \sum_i \mu_i^I H_i^I - \beta \sum_{i=1}^3 \omega_i(j_i + R) \right]
  \]

- [Di Pietro-Komargodski] conjectured a Cardy formula for the Cardy limit \( \beta \to 0 \) (based on free field examples):
  \[
  \log \mathcal{Z}_{S_1^6 \times S_5} = - \frac{\pi}{\omega_1 \omega_2 \omega_3} \left[ \frac{\kappa_1}{360} \left( \frac{2\pi}{\beta} \right)^3 + \frac{(\omega_1^2 + \omega_2^2 + \omega_3^2)(\kappa_2 - 3\kappa_3/2)}{72} \left( \frac{2\pi}{\beta} \right) \right. \\
  + \left. \frac{(\omega_1 + \omega_2 + \omega_3)^2 \kappa_3}{48} \left( \frac{2\pi}{\beta} \right) + \frac{\mu_2^I \kappa_f^{G_f}}{24} \left( \frac{2\pi}{\beta} \right) \right] + O(\beta^0, \log \beta)
  \]

- \( \kappa_i \) fixed by the perturbative anomalies
  \[
  \kappa_1 = -40\gamma - 10\delta, \quad \kappa_2 - \frac{3}{2} \kappa_3 = 16\gamma - 2\delta, \quad \kappa_3 = -2\beta, \quad \kappa_f^{G_f} = -48\mu^{G_f}.
  \]

- Anomaly coefficients in anomaly polynomial 8-form:
  \[
  l_8 = \frac{1}{4!} \left[ \alpha c_2(SU(2)_R)^2 + \beta c_p + \gamma p_1 + \delta p_2 \right] + \mu^{G_f} p_1 c_2(G_f)
  \]
Status of the Cardy formulae

Features:

- The SUSY Cardy formulae are completely determined by 't Hooft anomaly coefficients \((k \leftrightarrow (c - a))\) in 4d, \(\alpha, \beta, \gamma, \delta, \mu^{Gf}\) in 6d
- Universal background dependence in 4d/6d on background fields/metric

Status:

4d: [Di Pietro-Komargodski] derive the 4d formula based on Lagrangian theories (\(i.e.\) there exists a point in the space of continuous couplings where the theory becomes free)

⊕ Checks for Lagrangian theories from localization

[Di Pietro-Komargodski; Ardehali-Liu-Szepietowski; Ardehali; Di Pietro-Honda; ...], including the Schur index [Ardehali; Buican-Nishinaka]

However, \(\exists\) many non-Lagrangian theories:

- Formula holds based on non-Lagrangian examples [Buican-Nishinaka]
- Studied in the setting of “2d chiral algebra \(\leftrightarrow\) 4d \(\mathcal{N} = 2\) correspondence”\(^1\), where Schur index = VOA vacuum character [Beem-Rastelli] (\(\rightarrow a_{4d}\) in terms of 2d data).

6d: [Di Pietro-Komargodski] conjectured formula based on free multiplets

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\(^{1}\) [Beem-Lemos-Liendo-Peeaers-Rastelli-van Rees]
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6d $\mathcal{N} = (2, 0)$ check

- In the limit $\omega_1 = \omega_2 = \omega_3 = 1$ (unsquashed) and $\mu_f = 1$, 6d $\mathcal{N} = (2, 0)$ index computed via localization of 5d maximally sYM in $[\text{Kim}^2; \text{Kim}^3]$
- Compare with vacuum character of $\mathcal{W}_g$-algebra $[\text{Beem-Rastelli-van Rees}]$: ✔
- Type-$g$, $g = ADE$, 6d $\mathcal{N} = (2, 0)$ theory:

$$Z_{S^1 \times S^5}^g = e^{\frac{\beta}{6} h^\vee_g |g| \left( \frac{\beta}{2\pi} \right)^{\frac{r_g}{2}} \prod_{\alpha \in \Delta^g_+} \left( 1 - e^{-\beta (\alpha \cdot \rho_g)} \right) \eta \left( e^{-\frac{4\pi^2}{\beta}} \right)^{r_g}}$$

- In the Cardy limit, the $\eta$-factor becomes the dominant contribution, and we find

$$\log Z_{S^1 \times S^5}^g = \frac{r_g \pi^2}{6\beta} + \mathcal{O}(\beta^0, \log \beta)$$

- Compare to proposal:

$$\log Z_{S^1 \times S^5}^g = \left. \frac{r_g \pi^2 (2\omega_1 \omega_2 + 2\omega_2 \omega_3 + 2\omega_3 \omega_1 + \mu_f^2 - \omega_1^2 - \omega_2^2 - \omega_3^2)}{24\beta \omega_1 \omega_2 \omega_3} \right|_{\omega_i = 1, \mu_f = 1} + \mathcal{O}(\beta^0, \log \beta) : \ ✔$$
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Summary and outlook
We study a universal \((d - 1)\)-dimensional effective action, \(EA_{d-1}\), from dimensional reduction of SCFT\(_d\) on \(S^1_\beta\) in the limit \(\beta \to 0\):

I. Effective action:

\[
SUSY \ EA_{d-1} \sim \sum_i \kappa_i \int_{\mathcal{M}_{d-1}} \left[ A_{KK} \wedge (\cdots)_i + (SUSY - completion) \right] =: I_{A(\cdots)_i}
\]

a.) ● Thermal exp w/ SUSY bc [Banerjee-Bhattacharya-Bhattacharyya-Jain-Minwalla-Sharma]:

- **gauge-invariant** \(\oplus\) anomaly-induced \((\sim O(\beta))\) \(\oplus\) non-local \((\beta - indep)\)

- Classification of \((d-1)\)-dim counter terms (in 3d [Closset-Dumitrescu-Festuccia-Komargodski], in 5d [Chang-MF-Lin-Wang])

b.) Finite: no counterterms in \(d\)-dimensions

c.) Non-renormalized: insensitive to changes in couplings/moduli (promote \(\kappa_i\) to bg field \(\rightsquigarrow\) non-gauge invariant & \(\nabla\) anomaly to soak up non-gauge invariance \(\frac{1}{\epsilon}\))^2

II. Evaluate on SUSY background \(\rightsquigarrow\) SUSY CS invariants \(\leftrightarrow\) Geometric invariants

III. Determine \(\kappa_i\) from **weakly-coupled phase** (KK reduction of free fields)

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^2 Alternative argument: relation to global anomalies + global anomaly matching (see later)
Two comments:

1. We are only dealing with SCFTs in the following.

2. None of the subsequent arguments are reliant on (any type of) localization.

3. In (e.g.) 4d the arguments for the Cardy limit is strictly only valid when the non-local contributions from the holonomies of 4d vector multiplets are suppressed in the 3d effective action (i.e. runaway dof’s are suppressed!) → see e.g. [Di Pietro-Honda].
   - On the Higgs branch in 4d/6d, there are no vector multiplets and thus runaway dof’s should not pose a problem
   - Furthermore, in e.g. 4d $\mathcal{N} = 2$, if the theory has a pure HB, $c - a \propto \text{dim HB} > 0$, while in it was argued that such issues only appear for $a - c < 0$ [Di Pietro-Honda].
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3d Chern-Simons Effective Action:

- 4d Metric:

\[ ds_4^2 = \left( d\tau + \frac{\beta}{2\pi} A_i dx^i \right)^2 + h_{ij} dx^i dx^j \]

- 3d Chern-Simons Effective Action (c.f. [Closset-Dumitrescu-Festuccia-Komargodski; Di Pietro-Komargodski]):

\[
- \log Z = iW_{\text{eff}} + O(\beta^0, \log \beta),
\]

where the graviphoton \( A \) is of order \( O(\beta^{-1}) \) and \( V_R \) the R-symmetry background gauge field.
6d Metric:
\[
\text{d}s_6^2 = \left(\text{d}\tau + \frac{\beta}{2\pi} A_i \text{d}x^i\right)^2 + h_{ij} \text{d}x^i \text{d}x^j
\]

5d Chern-Simons Effective Action:
\[
iW_{\text{eff}} = \frac{i}{8\pi^2} \left(\frac{\kappa_1}{360} I^{\text{AFF}} + \frac{\kappa_2 - \frac{3}{2}\kappa_3}{144} I^{\text{ARR}} - \frac{\kappa_3}{24} I^{\text{AF}_R F_R} - \frac{\kappa_f G_f}{24} I^{\text{AF}_f F_f}\right),
\]
\[-\log Z = iW_{\text{eff}} + \mathcal{O}(\beta^0, \log \beta),
\]

where the “counterterms” (classified in [Chang-MF-Lin-Wang]) are
\[
I^{\text{AFF}} \equiv \int A \wedge dA \wedge dA + \text{SUSY completion},
\]
\[
I^{\text{ARR}} \equiv \int A \wedge \text{tr} (R \wedge R) + \text{SUSY completion},
\]
\[
I^{\text{AF}_R F_R} \equiv \int A \wedge \text{Tr} (F_R \wedge F_R) + \text{SUSY completion},
\]
\[
I^{\text{AF}_f F_f} \equiv \int A \wedge \text{Tr} (F_{G_f} \wedge F_{G_f}) + \text{SUSY completion}.
\]

Here, \(A\) is the \(U(1)_{\text{KK}}\) graviphoton (which in the \(\beta \to 0\) limit scales as \(\beta^{-1}\)), \(R\) denotes the Riemann curvature 2-form of the 5d background metric \(h_{ij}\).
Question:
6d/4d non-Lagrangian theories: Can we go to a free field description without spoiling the $\beta \to 0$ limit?

Idea:
Go out onto the moduli space!
Effective actions

6d/4d SCFT
e.g. \((A_1, A_2) = AD_{N_f=1}(SU(2))\)

Turn on VEV: \(\langle \Phi \rangle \sim m_{\text{moduli}}\)

6d/4d Wilsonian effective action
Cutoff \(\Lambda\): \(m_{\text{moduli}} < \Lambda, E < \Lambda\)
e.g. BPS quiver: \(\bigcirc \rightarrow \bigcirc\)

\[ \int \text{out massive fields} \]

reduce on \(S^1\)
\[ \int \text{out massive fields} \]

6d/4d effective action:
\(E < m_{\text{moduli}}\)
e.g. free ml vm

5d/3d Wilsonian effective action
Cutoff \(\Lambda\): \(m_{\text{moduli}} < \Lambda < \beta^{-1}, E < \Lambda\)

\[ \int \text{out } ml \text{ fields} \]

\[ \int \text{out } ml \text{ fields} \]

5d/3d effective action
\(\kappa_1 A \wedge dA \wedge dA + \cdots\)
Contributions from massive BPS particles:

From central extension of superconformal algebra or (equivalently) from gauge fixing conformal supergravity classify BPS objects:

★ 4d $\mathcal{N} = 2$:

**HB**: BPS strings: $\exists$ weakly-coupled phase (hK $\sigma$-model)
★ **Upshot**: Do not affect $\beta \to 0$ $\text{EA}_{3d}$ (from localization)$^3$

**CB**: BPS particles: $\nexists$ weakly-coupled phase i.e. ($\exists$ mutually non-local BPS particles)
★ **“Down”shot**: They affect $\beta \to 0$ $\text{EA}_{3d}$ (e.g. IR formula [Cordova-Shao])

**Mixed**: BPS particles, strings and domain-walls: $\nexists$ weakly-coupled phase

★ 6d $\mathcal{N} = (1, 0)$:

**HB**: BPS codim-2 branes: $\exists$ weakly-coupled phase (hK $\sigma$-model)
★ **Upshot**: Do not affect $\beta \to 0$ $\text{EA}_{5d}$ (infinite energy on $S^1 \times \mathbb{R}^5$)$^2$

**TB**: BPS strings: $\exists$? weakly-coupled phase
★ **“Down”shot**: They affect $\beta \to 0$ $\text{EA}_{5d}$

**Mixed**: BPS strings and codim-2 branes: $\exists$? weakly-coupled phase

---

$^3$Alternative argument: e.g. in 4d: $c - a = \text{cst}$ on HB and $\kappa = \text{cst} \Rightarrow$ both captured from weakly-coupled description. Same logic applies in 6d.
Effective actions (again)

6d/4d SCFT

Turn on VEV: $\langle \Phi \rangle \sim m_{\text{moduli}}$

6d/4d Wilsonian effective action
Cutoff $\Lambda$: $m_{\text{moduli}} < \Lambda$, $E < \Lambda$

∫ out massive fields
Reduce on $S^1$
∫ out massive fields

6d/4d effective action:
$E < m_{\text{moduli}}$

5d/3d Wilsonian effective action
Cutoff $\Lambda$: $m_{\text{moduli}} < \Lambda < \beta^{-1}$, $E < \Lambda$

∫ out ml fields
Reduce on $S^1$
∫ out all fields

5d/3d effective action:
$\kappa_1 A \wedge dA \wedge dA + \cdots$
Conclusion:

I. The SUSY effective action $W$ in $(d - 1)$-dimensions is meaningful on the moduli spaces and captures the high-temperature behavior of the SCFT on $S^1 \times S^{d-1}$.

II. On HB the $\beta \to 0$ limit not affected upon integrating out massive BPS objects.

Remains to determine ingredients:

i) Determine geometric invariants $I_j$ (= SUSY CS terms = higher-derivative terms in 5d/3d)

ii) Determine the CS coefficients $\kappa_j$ from free fields in $d$-dimensions (on Higgs branch)
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Geometric invariants

(classical) Chern-Simons terms evaluated on closed manifold $\leadsto$ topological invariant

(classical) supersymmetric Chern-Simons terms evaluated on supergravity background of a closed manifold $\leadsto$ geometric invariant
Geometric invariants & their evaluation on squashed $S^5$:

$$EA: \quad i W_{\text{eff}} = \frac{i}{8\pi^2} \left( \frac{\kappa_1}{360} I_{\text{AFF}} + \frac{\kappa_2 - \frac{3}{2}\kappa_3}{144} I_{\text{ARR}} - \frac{\kappa_3}{24} I_{\text{AF}_R F_R} - \frac{\kappa^G_f}{24} I_{\text{AF}_F F_f} \right)$$

<table>
<thead>
<tr>
<th>Action:</th>
<th>$I_{\text{AFF}}$</th>
<th>$I_{\text{ARR}}$</th>
<th>$I_{\text{AF}_R F_R}$</th>
<th>$I_{\text{AF}_F F_f}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>KK vm action</td>
<td>related to higher derivative</td>
<td>$-4 \left[ WW - \frac{1}{6} RR \right]$</td>
<td>related to higher derivative</td>
<td>$-\frac{1}{2} RR$</td>
</tr>
<tr>
<td>$SW \oplus V_{\text{KK}}$</td>
<td>$SW \oplus V_{\text{KK}}$ and $P_{\text{KK}}$</td>
<td>$P_{\text{KK}}$</td>
<td>$SW \oplus V_{\text{KK}} \oplus (V_f)^2$</td>
<td></td>
</tr>
</tbody>
</table>

- **5d Standard Weyl $SW$**: $SW = (g_{\mu\nu}, D, V_{ij}^{\alpha}, \nu_{\mu\nu}, b_{\mu}, \psi_i^j, \chi^i)$
- **5d Vector multiplet $V$**: $V = (W_\mu, M, \Omega^i_\alpha, Y_i^j)$ – 2 different ones:
  - $V_{\text{KK}}$: $W_{\text{KK}} \propto$ (graviphoton) and $M_{\text{KK}} \sim$ (warping factor)
  - $V_f$: $M_f = \text{constant} (= \text{flavor mass})$
- **5d Linear multiplet $L$** (compensator): $L = (L_{ij}, \varphi^i_\alpha, E^\mu, N)$
- **5d Poincaré $P_{\text{KK}}$**: Gauge-fixing $SW$ with $V$ and $L$ compensators: $P_k = \frac{SW \oplus V_{\text{KK}} \oplus L}{(\text{Gauge fixing})_k}$

"KK gauge": compensator $\hat{V}$ fixed to be $V_{\text{KK}}$, i.e.

$$\hat{L}_j^i = \frac{1}{2} \hat{L}(\sigma_3)^i_j, \quad \hat{L} = 1, \quad b_\nu = 0, \quad \phi^i = 0.$$
Interlude: Rigid SUSY on $\mathcal{M}_5$

Rigid SUSY for Standard Weyl [Alday, Benetti, MF, Richmond, Sparks] and vector/linear/gauge-fixed Poincaré [MF, unpublished]:

$\iff$ We obtain that $(M_5, g)$ is equipped with a conformal Killing vector generating a transversally holomorphic foliation (THF). The transverse metric $g_4$ is an arbitrary Hermitian metric with respect to the transverse complex structure $+$ explicit equations for background fields.

Some examples:
- Product metric $M_5 = \mathbb{R} \times M_4$ or $M_5 = S^1 \times M_4$, where $M_4$ is Hermitian, Sasakian, $S^1 \times S^4$, Twisted indices, etc

Upshot:
- Squashed sphere partition function/geometric SUSY invariants only dependent on geometric quantities (=squashing parameters $\omega_i$) parametrizing Killing vector $\xi = \sum_{i=1}^{3} \omega_i \partial \phi_i$

Conjecture:
- SUSY invariants/partition functions/... are only dependent on the THF.
Geometric invariants & their evaluation on squashed $S^5$:

We now fix (for simplicity) the space to be $S^1 \times S^5_{\text{sq}}$, with the 6d metric:

$$\text{ds}^2_{S^1 \times S^5} = r_5^2 \sum_{i=1}^{3} \left[ dy_i^2 + y_i^2 \left( d\phi_i + \frac{ia_i}{r_5} d\tau \right)^2 \right] + d\tau^2 ,$$

The corresponding 5d metric is then:

$$\text{ds}^2_{S^5} = \sum_{i=1}^{3} (dy_i^2 + y_i^2 d\phi_i^2) + \tilde{\kappa}^{-2} \mathcal{Y}^2 ,$$

$$\mathcal{Y} = \tilde{\kappa}^2 \sum_{i=1}^{3} a_i y_i^2 d\phi_i , \quad \tilde{\kappa}^{-2} = 1 - \sum_{j=1}^{3} y_j^2 a_j^2 ,$$

where now

$$A = m_{KK} r_5 \mathcal{Y} , \quad m_{KK} = \frac{2\pi i}{\beta} .$$

and we now proceed to the evaluation of $I_{AFF}, \ldots, I_{AF_f F_f}$ in this background.
Geometric invariants & their evaluation on squashed $S^5$:

$$iW_{\text{eff}} = \frac{i}{8\pi^2} \left( \frac{\kappa_1}{360} I_{\text{AFF}} + \frac{3}{2} \frac{\kappa_2}{144} I_{\text{ARR}} - \frac{\kappa_3}{24} I_{\text{AF}F_R} - \frac{\kappa_{Gf}}{24} I_{\text{AF}f_F} \right)$$

$$I_{\text{AFF}}, I_{\text{AF}f_F}^{Gf} \equiv \int A \wedge dA \wedge dA + \text{SUSY completion}$$

$$\equiv \text{SUSY Euclidean vm's, } \{\mathcal{V}_I\}_I, \text{ action coupled to } S\mathcal{W}$$

$$= S_{\text{vm}} (\mathcal{V}_I, \mathcal{V}_J, \mathcal{V}_K)$$

$$= \int_{\mathcal{M}_5} c_{IJK} \left[ \frac{1}{2} W^I \wedge F^J(W) \wedge F^K(W) - \frac{3}{2} M^I F^J(W) \wedge *F^K(W) \right.$$

$$+ \frac{3}{2} M^I dM^J \wedge *dM^K - 3 M^I M^J \left( 2F^K(W) + M^K \nu \right) \wedge *\nu$$

$$+ M^I \left( 3(Y^J)_{ij}(Y^K)^{ij} + \frac{1}{4} M^J M^K \left[ \frac{R}{2} - D \right] \right) \text{vol}_5 \right]$$
Geometric invariants & their evaluation on squashed $S^5$:

$$i W_{\text{eff}} = \frac{i}{8\pi^2} \left( \frac{\kappa_1}{360} I_{\text{AFF}} + \frac{\kappa_2 - \frac{3}{2}\kappa_3}{144} I_{\text{ARR}} - \frac{\kappa_3}{24} I_{\text{AFFR}} - \frac{\kappa_f^G}{24} I_{\text{AFF}_f} \right)$$

$$I_{\text{AFF}} \equiv \int A_{\text{KK}} \wedge dA_{\text{KK}} \wedge dA_{\text{KK}} + \text{SUSY completion}$$

$$\equiv \text{SUSY Euclidean vm, } \mathcal{V}_{\text{KK}}, \text{ action coupled to } S\mathcal{W}$$

$$= S_{\text{vm}} (\mathcal{V}_{\text{KK}}, \mathcal{V}_{\text{KK}}, \mathcal{V}_{\text{KK}})$$

$$= \frac{m_{\text{KK}}^3}{2} \int_{\mathcal{M}_5} \eta \wedge d\eta \wedge d\eta + \int_{\mathcal{M}_5} d* (\cdots)$$

$$= \frac{m_{\text{KK}}^3}{2} \left(2\pi\right)^3 \frac{(2\pi)^3}{\omega_1\omega_2\omega_3}$$
Geometric invariants & their evaluation on squashed $S^5$:

\[ \text{EA: } i W_{\text{eff}} = \frac{i}{8\pi^2} \left( \frac{\kappa_1}{360} I_{\text{AFF}} + \frac{\kappa_2 - \frac{3}{2}\kappa_3}{144} I_{\text{ARR}} - \frac{\kappa_3}{24} I_{\text{AFRF}} - \frac{\kappa_f G_f}{24} I_{\text{AFfFf}} \right) \]

\[ I_{\text{AFfFf}}^G \equiv \int A_f \wedge dA_f \wedge dA_{\text{KK}} + \text{SUSY completion} \]

\[ \equiv \text{SUSY Euclidean vms, } \{ V_{\text{KK}}, V_f, V_f^\ell \}, \text{ action coupled to } S\mathcal{W} \]

\[ = S_{\text{vm}} (V_{\text{KK}}, V_f, V_f) \]

\[ = \frac{m_{\text{KK}}}{2} \int_{\mathcal{M}_5} \eta \wedge d\eta \wedge d\eta + \int_{\mathcal{M}_5} d * (\cdots) \]

\[ = \frac{m_{\text{KK}}}{2} \frac{(2\pi)^3}{\omega_1 \omega_2 \omega_3} \mu_f^2 \]
Geometric invariants & their evaluation on squashed $S^5$:

\[ \mathbf{EA:} \quad iW_{\text{eff}} = \frac{i}{8\pi^2} \left( \frac{\kappa_1}{360} I_{\text{AFF}} + \frac{\kappa_2 - \frac{3}{2}\kappa_3}{144} I_{\text{ARR}} - \frac{\kappa_3}{24} I_{\text{AF}_R F_R} - \frac{\kappa_f^G}{24} I_{\text{AF}_F F_f} \right) \]

Remains \( I_{\text{ARR}} = \int A \wedge \text{tr} (R \wedge R) + \text{SUSY} \) & \( I_{\text{AF}_R F_R} = \int A \wedge \text{Tr} (F_R \wedge F_R) + \text{SUSY} \).

- Both related to higher derivative terms by SUSY
- There are 3 higher-derivative terms: \( R^2 \), Weyl\(^2 \) and Riemann\(^2 \) & \( \exists \) SUSY completion \([\text{Hanaki-Ohashi-Tachikawa; Ozkan-Pang; Butter-Kuzenko-Novak-Tartaglino-Mazzucchelli}]\)

- Field redefinitions \( \implies \) 1 linear combination trivial \( \implies \) only 2 independent ones:
  - **WW:** superconformal, \( V \oplus SW \) \([\text{Hanaki-Ohashi-Tachikawa}]\)
  - **RR:** Poincaré, \( \mathcal{P}_{\text{KK}} = SW \oplus V \oplus \mathcal{L}/\text{KK gauge} \) \([\text{Ozkan-Pang}]\)

- Wick rotate
- (Tediously) Evaluate on appropriate solution

Then:
\[ I_{\text{ARR}} \equiv -4 \left[ \text{WW} - \frac{1}{6} \text{RR} \right], \]
\[ I_{\text{AF}_R F_R} \equiv -\frac{1}{2} \text{RR} \]
Geometric invariants & their evaluation on squashed $S^5$:

**EA:**

$$iW_{\text{eff}} = \frac{i}{8\pi^2} \left( \frac{\kappa_1}{360} I_{\text{AFF}} + \frac{\kappa_2 - \frac{3}{2}\kappa_3}{144} I_{\text{ARR}} - \frac{\kappa_3}{24} I_{\text{AF}_R F_R} - \frac{\kappa_{f}^G}{24} I_{\text{AF}_f F_f} \right)$$

$$I_{\text{AF}_R F_R} \equiv \int_{M_5} A \wedge \text{tr} (F_R \wedge F_R) + \text{SUSY completion}$$

$$\equiv -\frac{1}{2} \left( \text{SUSY Euclidean RR action in terms of } \mathcal{P}_{KK} \right)$$

$$= \text{long long expression}$$

$$= \frac{(2\pi)^3}{2} \frac{(\omega_1 + \omega_2 + \omega_3)^2}{\omega_1 \omega_2 \omega_3} m_{KK}$$
Geometric invariants & their evaluation on squashed $S^5$:

$$EA: \quad iW_{\text{eff}} = \frac{i}{8\pi^2} \left( \frac{\kappa_1}{360} I_{\text{AFF}} + \frac{\kappa_2 - \frac{3}{2} \kappa_3}{144} I_{\text{ARR}} - \frac{\kappa_3}{24} I_{\text{AF}_R\text{FR}} - \frac{\kappa_f^G}{24} I_{\text{AF}_f\text{F}_f} \right)$$

$$I_{\text{ARR}} \equiv \int_{M_5} A \wedge \text{tr} (R \wedge R) + \text{SUSY completion}$$

$$\equiv 4 \left( \text{SUSY Euclidean WW action in terms of } SW \oplus V_{KK} \right) + \frac{2}{3} \left( \text{SUSY Euclidean RR action in terms of } P_{KK} \right)$$

$$= -\frac{2}{3} \frac{(\omega_1 + \omega_2 + \omega_3)^2}{\omega_1 \omega_2 \omega_2} (2\pi)^3 m_{KK}$$

$$+ 4 \left[ \frac{\omega_1^2 + \omega_2^2 + \omega_3^2}{2\omega_1 \omega_2 \omega_3} - \frac{(\omega_1 + \omega_2 + \omega_3)^2}{6\omega_1 \omega_2 \omega_3} \right] (2\pi)^3 m_{KK}$$

$$= \frac{(2\pi)^3}{2} \frac{(\omega_1 + \omega_2 + \omega_3)^2}{\omega_1 \omega_2 \omega_3} m_{KK}$$
Summary:

\[ iW_{\text{eff}} = \frac{i}{8\pi^2} \left( \frac{\kappa_1}{360} I_{\text{AFF}} + \frac{\kappa_2 - \frac{3}{2}\kappa_3}{144} I_{\text{ARR}} - \frac{\kappa_3}{24} I_{\text{AFR}F_R} - \frac{\kappa_f^{G_f}}{24} I_{\text{AFf}F_f} \right) \]

where

\[ m_{\text{KK}} = \frac{2\pi i}{\beta}, \]

we find:

\[ I_{\text{AFF}} = \int_M A \wedge dA \wedge dA + \text{SUSY completion} = \frac{(2\pi)^3}{270 \omega_1 \omega_2 \omega_3} \left( \frac{2\pi i}{\beta} \right)^3 \]

\[ I_{\text{ARR}} = \int_M A \wedge \text{tr} (R \wedge R) + \text{SUSY completion} = -2(2\pi)^3 \frac{(\omega_1^2 + \omega_2^2 + \omega_3^2)}{\omega_1 \omega_2 \omega_3} \left( \frac{2\pi i}{\beta} \right) \]

\[ I_{\text{AFR}F_R} = \int_M A \wedge \text{Tr} (F_R \wedge F_R) + \text{SUSY completion} = \frac{(2\pi)^3}{2} \frac{(\omega_1 + \omega_2 + \omega_3)^2}{\omega_1 \omega_2 \omega_3} \left( \frac{2\pi i}{\beta} \right) \]

\[ I_{\text{AFf}F_f} = \int_M A \wedge \text{Tr} (F_{G_f} \wedge F_{G_f}) + \text{SUSY completion} = \frac{(2\pi)^3}{\omega_1 \omega_2 \omega_3} \mu_f^2 \left( \frac{2\pi i}{\beta} \right) \]

\[ \log Z_{S^1 \times S^5} = -\frac{\pi}{\omega_1 \omega_2 \omega_3} \left[ \frac{\kappa_1}{360} \left( \frac{2\pi}{\beta} \right)^3 + \frac{(\omega_1^2 + \omega_2^2 + \omega_3^2)(\kappa_2 - \frac{3}{2}\kappa_3)}{72} \left( \frac{2\pi}{\beta} \right) \right. \]

\[ \left. + \frac{(\omega_1 + \omega_2 + \omega_3)^2 \kappa_3}{48} \left( \frac{2\pi}{\beta} \right) + \frac{\mu_f^2 \kappa_f^{G_f}}{24} \left( \frac{2\pi}{\beta} \right) \right] + \mathcal{O}(\beta^0, \log \beta), \]
Some remarks on SUSY CS terms:

- Dependent on THF (complex structure on $S^1 \times S^5$)! $\Rightarrow$ Found to be true for sphere-type topology (trivial fundamental group).

- Independent of choice of solution (e.g. RR-term independent of choice of gauge-fixing, independent of choice of vector multiplet solution, etc.).

- In some cases (FFF) we can prove that for any SUSY geometry, $S^1 \times M_5$, terms reduce to contact volume $\int \eta \wedge d\eta \wedge d\eta$. 
Motivation: 2d Cardy formula

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- Preliminaries
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Some Remarks

Summary and outlook
KK reduction of free fields

- Assume weakly coupled phase in the EFT (e.g. Higgs branch)
- Compute triangle diagrams (and bubbles)
- 1-loop exact

\[ A \wedge dA \wedge dA \quad A \wedge \text{tr} (\mathcal{R} \wedge \mathcal{R}) \quad A \wedge \text{tr} (F_R \wedge F_R) \quad A \wedge \text{tr} (F_f \wedge F_f) \]

- Internal legs either massive 2-forms or massive fermions.
- Sum over KK tower contributions \( \sim \kappa_i \)
From KK-reduction of (anti)chiral fermions $\psi_{\pm}$ and self-dual 2-forms, $B$

[Bonetti-Grimm-Hohenegger]:

\[
\mathcal{I}_{\psi_-}^n = \frac{1}{48\pi^2} n^3 l_{AFF} + \frac{1}{384\pi^2} n l_{ARR},
\]

\[
\mathcal{I}_{\psi_+}^n = -\frac{1}{48\pi^2} n^3 l_{AFF} - \frac{1}{384\pi^2} n l_{ARR},
\]

\[
\mathcal{I}_B^n = -\frac{4}{48\pi^2} n^3 l_{AFF} + \frac{8}{384\pi^2} n l_{ARR},
\]

Thus, for 6d $\mathcal{N} = (1, 0)$ supermultiplets (tensor multiplet $T \ni \{B, \psi^i_\pm\}$, vm $V \ni \{\psi^i_\pm\}$, hm $H \ni 2\{\psi^i_\pm\}$)

\[
\mathcal{I}_T = \frac{2 - 4}{48\pi^2} \frac{1}{120} l_{AFF} - \frac{2 + 8}{384\pi^2} \frac{1}{12} l_{ARR} - \frac{2}{32\pi^2} \frac{1}{12} l_{AF_R F_R}
\]

\[
\mathcal{I}_V = \frac{-2}{48\pi^2} \frac{1}{120} l_{AFF} - \frac{-2}{384\pi^2} \frac{1}{12} l_{ARR} - \frac{-2}{32\pi^2} \frac{1}{12} l_{AF_R F_R}
\]

\[
\mathcal{I}_{H}^{nH} = \frac{2}{48\pi^2} \frac{1}{120} l_{AFF} - \frac{2}{384\pi^2} \frac{1}{12} l_{ARR} - \frac{2}{32\pi^2} \frac{1}{12} l_{AF_f F_f}
\]

\[
\mathcal{I}_{H}^{USp(2n_H)} = \frac{1}{48\pi^2} \frac{1}{120} l_{AFF} - \frac{1}{384\pi^2} \frac{1}{12} l_{ARR} - \frac{1}{32\pi^2} \frac{1}{12} l_{AF_f F_f}
\]
Contributions from KK reduction

Thus, we conclude that from KK reduction of massless 6d fields in the weakly coupled phase (vortex-sheets do not contribute local CS terms)

\[- \log Z = \frac{i}{8\pi^2} \left( \frac{n_T + n_V - n_H}{360} I_{AFF} + \frac{n_H + 5n_T - n_V}{288} I_{ARR} + \frac{n_T - n_V}{24} I_{AF_RF_R} + \frac{1}{24} I_{USp(2n_H)} \right) + \mathcal{O}(\beta^0, \log \beta).\]
Motivation: 2d Cardy formula

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Some Remarks

Summary and outlook
Matching to perturbative anomalies on the Higgs branch

- Anomaly matching on the Higgs branch [Shimizu-Tachikawa-Zafrir] gives us $\text{AP}_{\text{HB}}$ on Higgs branch.

- Can identify anomaly coefficients of $\text{AP}_{\text{SCFT}} \alpha, \beta, \gamma, \delta$ in terms of free hypers on HB.$^4$

- By our arguments, there are no corrections to $-\log Z_{\text{SCFT}}$ when going on Higgs branch, i.e. $\mathcal{W}_{\text{SCFT}} \sim \mathcal{W}_{\text{HB}}$ (careful about symmetry breaking).

- Identify:

  \[
  \begin{align*}
  \kappa_1 &= 60\delta, \\
  \kappa_2 - \frac{3}{2}\kappa_3 &= -30\delta \\
  \kappa_3 &= -2\beta, \\
  \kappa_f &= -48\mu.
  \end{align*}
  \]

- Consistent with

  \[
  \begin{align*}
  \kappa_1 &= -40\gamma - 10\delta, \\
  \kappa_2 - \frac{3}{2}\kappa_3 &= 16\gamma - 2\delta, \\
  \kappa_3 &= -2\beta, \\
  \kappa_f &= -48\mu.
  \end{align*}
  \]

$^4$Recall $\gamma = -\frac{7}{4}\delta$ on HB.
1 Motivation: 2d Cardy formula

2 Higher-dimensional Cardy formulae
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3 Proof:
   • Overview
   • Effective actions
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4 Global anomalies from effective action

5 Some Remarks

6 Summary and outlook
4d $\mathcal{N} = 2$ examples

Higgs branch EA of 4d $\mathcal{N} = 2$ theories:

(i) $\{\text{free ml hypermultiplets}\} \equiv \text{pure Higgs branch}$.
(ii) $\{\text{free ml hypermultiplets}\} \oplus \{\text{free ml } U(1) \text{ vector multiplets}\}$.
(iii) $\{\text{free}\} \oplus \{\text{decoupled interacting SCFT (with trivial HB)}\}$.

Examples:

(i) AD theories (non-Lagrangian)
   a) $(A_1, A_{2n+1})$: $\dim\text{CB} = n$, $\dim_{\mathbb{H}}\text{HB} = 2$, and $\kappa \sim c - a = \frac{1}{24}$
   b) $(A_1, D_{2n+2})$: $\dim\text{CB} = n$, $\dim_{\mathbb{H}}\text{HB} = 1$, and $\kappa \sim c - a = \frac{2}{24}$

(ii) Lagrangian theories with $n_v$ vectors and $n_h$ hypers $\Rightarrow \kappa \sim c - a = \frac{n_h - n_v}{24}$

(iii) $(A_1, D_{2n+1}) \leadsto (A_1, A_{2n-2}) \oplus \{\text{free hyper}\}$:

$$\kappa \sim (c - a)[(A_1, D_{2n+1})] = (c - a)[(A_1, A_{2n-2})] + (c - a)[\text{free hyper}] = 1/24$$
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Some Remarks

Summary and outlook
Global anomalies from CS effective action

∃ direct relation between the CS levels $\kappa_i$ and the phases from global anomalies:

- Consider spacetime $S^1 \times S^1_{x_5} \times \mathcal{M}_4$:
  \[
  ds_6^2 = \left( d\tau + \frac{\beta}{2\pi} A_i dx^i \right)^2 + dx_5^2 + h^{(4)}_{ij} dx^i dx^j, \quad x_5 \sim x_5 + 2\pi R_5.
  \]

- Large (bg) diffeomorphism (preserving bc for the fermionic dofs along $S^1_{x_5}$):
  \[
  \tau \to \tau + n\frac{\beta}{2\pi R_5} x_5, \quad (n \in \mathbb{Z})
  \]
  ➞ background large gauge transformation of the graviphoton $A$
  \[
  A \to A + \frac{n}{R_5} dx^5
  \]

- For theories with (mixed) gravitational anomalies, the partition function $Z$ is not invariant under such a large bg diffeomorphism:
  \[
  Z[A + \delta A] = e^{-i\pi \eta} Z[A].
  \]

- Global gravitational anomalies ⇔ anomalies under large gauge transformations
Global anomalies from CS EA

The 5d CS EA captures this anomalous diffeomorphism (related thd: [Golkar-Son, Golkar-Sethi, Chowdhury-David])

- Under
  \[ A \rightarrow A + \frac{n}{R_5} \, dx^5 \]
  
  the effective action transforms as
  \[
  \delta W_{\text{eff}} = n \int_{\mathcal{M}_4} \left( \frac{\kappa_1}{480\pi} \, dA \wedge dA - \frac{\pi}{72} (\kappa_2 - \frac{3}{2} \kappa_3) p_1 - \frac{\pi}{12} \kappa_3 c_2(SU(2)_R) - \frac{\pi}{12} \kappa_f^G c_2(G_f) \right)
  \]

- The integral satisfies quantization conditions on the manifold \(\mathcal{M}_4\)
  \[
  m_1 \equiv \frac{1}{2(2\pi)^2} \int_{\mathcal{M}_4} dA \wedge dA \in \mathbb{Z}, \quad m_2 \equiv \frac{1}{24} \int_{\mathcal{M}_4} p_1 \in 2\mathbb{Z}
  \]
  \[
  m_3 \equiv \int_{\mathcal{M}_4} c_2(SU(2)_R) \in \mathbb{Z}, \quad m_f \equiv \int_{\mathcal{M}_4} c_2(G_f) \in \mathbb{Z}
  \]

- We find that the anomalous phase in \(Z[A + \delta A] = e^{-i\pi \eta} Z[A]\) is given by (\(\eta\) invariant)
  \[
  \eta = \frac{nm_1}{60} \kappa_1 + \frac{2nm_2}{3} (\kappa_2 - \frac{3}{2} \kappa_3) - \frac{n}{12} (m_3 \kappa_3 + m_f \kappa_f^G) \quad \text{mod } 2
  \]
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Some Remarks

Summary and outlook
Some Remarks I:

I. $\kappa_i$ related to $\eta$:

It follows that $\kappa_i$ are fixed (up to possible jumps). Supports our argument that $\kappa_i = \text{cst}$ along moduli space flows.

II. Global gravitational anomalies given by $\eta$-invariant on Higgs branch:

We argue that $\kappa_i = \text{cst}$ on the Higgs branch, and that $\kappa_i$ are related to global gravitational anomalies. Thus, it follows that on the HB of an SCFT only the $\eta$-invariant contributes to global gravitational anomalies (up to possible jumps) [Hsieh-Tachikawa-Yonekura].

III. Anomaly matching on HB/TB:

Via $\kappa_j$ there exists a relation between (a type of) global anomalies and perturbative anomalies.
Some Remarks II:

IV. $a$ vs. $c$-anomalies

6d: $\kappa_i = \text{cst}$ on Higgs branch implies that going on the Higgs branch there are relations between $\Delta a$ and $\Delta c_i$ anomalies. [Cordova-Dumitrescu-Intriligator; ...].

4d: $\kappa = c - a = \text{cst}$ on Higgs branch $\implies \Delta a = \Delta c$ on Higgs branch. Follows also from anomaly-matching formula by [Shapere-Tachikawa].

V. Higher-derivative terms are geometric invariants

5d higher derivative terms (Chern-Simons terms $I_j$, $j = 1, \ldots, 4$) are geometric invariants only dependent on the THF of $\mathcal{M}_5$ (also independent of choice of gauge-fixing). Proven for $I_{AFF}$ and $I_{AFF_fF_f}$ + various checks [MF, unpublished].
Some Remarks III:
VI. “New Cardy limits” (for simplicity 4d)

**Brief primer:**

- **“New Cardy limit”** in 4d/6d [Choi-Kim-Kim-Nahmgoong; Cabo-Bizet-Cassani-Martelli-Murthy; Ardehali; Kim-Kim-Song; Nahmgoong; …] ↔ black hole microstates in dual sugra (in large \( N \) and \( \beta \omega_i^{(\text{here})} \ll 1 \) limit)\(^5\)

- **New features:** Complex fugacities (complex SUSY background/different spin structure; see also [Chang-MF-Lin-Wang]) \( \rightsquigarrow \) complex saddle points \( \rightsquigarrow \) additional leading order term \( \propto (5a - 3c) \neq 0 \) for holographic theories.

- **Different limit:** instead of \( \beta \omega_i^{(\text{here})} \rightarrow 0 \) (\( \equiv \)DK limit), one takes \( \beta \omega_i^{(\text{here})} \rightarrow 0 + i(\cdots) \rightsquigarrow \) different phases at large \( N \) and large-temperature (e.g. in \( \mathcal{N} = 4 \) sYM)! [Cabo-Bizet-Murthy; Ardehali-Hong-Liu]

\[ 5 \beta \omega_1^{(\text{here})} = \omega_1^{(\text{there})}, \beta \omega_2^{(\text{here})} = \omega_2^{(\text{there})}, \beta \omega_3^{(\text{here})} = 2\pi i + \omega_3^{(\text{there})} \]
Some Remarks III ctd:

V. “New Cardy limits” (for simplicity 4d)

“Effective action approach”:

- 3d effective action is different:

\[ W_{\text{theirs}} \sim CS_{\text{gauge-invariant}}^{\text{SUSY}} + W_{\text{gauge-non-invariant}}^{\text{SUSY}} \]

related to perturbative anomalies

\[ \sim \text{non-SUSY thermal effective action (thermal bc’s of fermions)} \]

\[ W_{\text{ours}} \sim CS_{\text{gauge-invariant}}^{\text{SUSY}} \]

related to global anomalies

- Arguments based on “thermal effective action”/“thermal AP”

[Jensen-Loganayagam-Yarom]

From our perspective:

- Would expect that \( CS_{\text{ours,SUSY}}^{\text{gauge-invariant}} + W_{\text{gauge-non-invariant}}^{\text{SUSY}} \) gives the right answer?

- Missing SUSY completion of “thermal effective action”, i.e.

\[ W_{\text{SUSY}}^{\text{gauge-non-invariant}} = ??? \quad [WIP] \]

Our general strategy should provide (independent) derivation!
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Some Remarks

Summary and outlook
Quick rundown

- Review proposals for Cardy formulae for 4d and 6d SCFTs
- Derived from (universal) effective action for non-Lagrangian theories!
- Captured by SUSY CS invariants/geometric invariants from the effective theory
- Prediction for universal high-temperature behavior of 4d/6d indices
- Relation to global gravitational anomalies from the effective theory
Quarantine homework:

I. Relation to modularity of 4d/6d index [Dedushenko-MF]: high-low temperature relation, e.g. Casimir vs Cardy.

II. Cardy formula on CB/TB: understand contributions from BPS strings and BPS states to the Cardy formula; e.g. recently global gravitational anomalies for p-form analyzed in [Hsieh-Tachikawa-Yonekura] → should give hint.

III. Understand SUSY EA for “new" C-Cardy limit in 4d/6d [Choi-Kim-Kim-Nahmgoong; Cabo-Bizet-Cassani-Martelli-Murthy; Kim-Kim-Song; Nahmgoong; ...] and relation to black hole microstates.

IV. Cardy limits in other (odd) dimensions from effective field theory and relation to global anomalies? Do we retain same type of universality?

V. Holographic dual of effective action? Higher derivative terms in 6d vs counter-terms. Measurable effect of superconformal anomalies [Chang-MF-Lin-Wang], · · ·

VI. Higher-derivative corrections to black hole entropy in AdS$_5$?
Thank You
For Your ATTENTION!

Feel free to contact me with questions/comments at:
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