May 28

Construction of Kuranishi structures

\[ X = \mathbb{K} \times \mathbb{U} \times (-\infty, 0) \times N_\mathbb{U} \cup \mathbb{C} \times N_\mathbb{U} \]

\( \mathbb{K} \) : moduli of Koba orbit of \( \mathbb{N}_\mathbb{U} \)

assume \( \text{BM} \)

\[ \hat{M}_{\text{g}, \text{e}, \text{h}, \text{e}}(X; E) \Rightarrow [E, E', E', \mathbb{U}, \mathbb{U}'] \]

\[ \mathbb{U} : \mathbb{E} \leftarrow \mathbb{U} \rightarrow \{ X \} \]

REH \( X \) comp. (modify strata replacing topological space fiber product by old fiber product)
Thm (Ishikawa)

\[ \hat{M}_{g,\ell_1,\ell_2} (X; E) \text{ has a k. str. with corners} \]

Explained in May 21

\[ \hat{M}_{g,\ell_1,\ell_2} (X; E) \supset \hat{M}_{g,\ell_1,\ell_2} (X; E) \]

\[ \text{ambient set} \]

\[ \left( \hat{z}, \hat{w}, \hat{z}', \hat{w}', \hat{z}'' \right) \]

Do not assume \( \psi \) to be holomorphic
Obstruction bundle data $\rightarrow$ Kuranishi structure

\[ P = \mathcal{M}_{g, \overline{e}, \overline{f}, \overline{E}}(X, E) \]

\[ \mathcal{X} \subset \mathcal{B}(P) \text{ codim } \nu \]

\[ \hat{F}_{g, \overline{e}, \overline{f}, \overline{E}}(X, E) \]

\[ \nu_{\mathcal{X}} \leq C^0(\overline{\mathcal{X}}, \mathcal{U}^{\mathcal{X}} \times \mathcal{U}^{\overline{\mathcal{X}}}) \]

or

\[ \mathcal{U}^{\mathcal{X}} \mathcal{T}(\mathbb{R} \times \mathcal{N}) \]

1. Repeat somewhere on $H$
2. Some aff. moves on $P$
3. Let $D_{\nu_{\mathcal{X}}} + E_{\nu_{\mathcal{X}}}(H) = C^0(\overline{\mathcal{X}}, \mathcal{U}^{\mathcal{X}} \times \mathcal{U}^{\overline{\mathcal{X}}}) \cap \mathcal{G}$
4) Invariant of $\Lambda\mathcal{A}(\pi)$.

Actually, we need more than just existence of Kuranishi structure.

1) Compatibility with boundary & current.

$\mathcal{O}$, $g \geq 0$, $l=0$, $l_t=1$

$\mathcal{H}^\sharp$
\[ e^{\frac{a}{b}} \quad (x, y) \rightarrow \left\{ \begin{array}{c}
\mathbb{R}^N \\ \text{if} \quad x \\
(\mathbb{R}^N + I) \quad \text{if} \quad y
\end{array} \right. \]
The Kuranishi structure of $M_{0,1}(X,E)$ restricts to the fiber product Kuranishi structure

$$\pi_t: M_{0,1}(X,E) \rightarrow R_0$$

is assumed to be weakly surjective

**Def.** \((X, \Delta) \xrightarrow{f} R\) is weakly sub.

\[ f \circ (\Delta_b) \xrightarrow{f_b} U_b \rightarrow R \text{ is a submersion.} \]
Lemma

\((X, \mathcal{U}) \to \mathbb{R} \quad \hat{f} \quad \mathcal{V} \)

\(f\) is weakly submersive, \(\mathcal{V}\) is smooth.

\(\quad \square \quad \text{fiber product} \quad (X, \mathcal{U}) \times_{\mathbb{R}} (\mathcal{V}, \mathcal{W}) \)

\(\quad \square \quad (u, \mathcal{Z}) \in X \times \mathbb{R} \)

\((U_p, \mathcal{L}) \quad \text{chart of } \mathcal{U} \text{ at } P \)

\((V_q, \mathcal{L}) \quad \text{chart of } \mathcal{V} \text{ at } Q \).
The $U_1 \times K$ would give

a $k$-chart.

Then added:

The Kuranishi structure of $\hat{M}_{g,2-2g+2}(X, \mathcal{F})$ can be chosen so that its restriction of the stack coincides with fiber product $k^*$ structure over $(R_g)^*$. 
Why we need it?

States: We want to integrate differential form $\omega$ and apply

$$\int_{\partial C} S \, \omega = S \cdot \int_{\partial C} \omega$$

Integration depends on $C$.

perturbation. States holds if $\partial C$ perturbations are compatible at $A$. 

\[ q \]
The \((X, \mathcal{U})\) space with \(X\), sth \((\partial X, \partial \mathcal{U})\) has \(C^1\)-perturbations which are compatible at the corners.

\[ \Rightarrow \text{We can extend it to a } C^1\text{-perturbations of } (X, \mathcal{U}). \]
Proof is not so easy.


Thus, we construct system of Kuranishi structures which are compatible at the boundary and corners.

If obstruction bundle data have such property

$\Rightarrow$ associated Kuranishi structure has this property.
I will go back to this point later.

Another point which is very important in SFT.

Not we have another compactification

\[ \tilde{M}_{\beta_1 \beta_2 \beta_3} (X, E) \]

\[ \Rightarrow \tilde{M}^{\rm pl}_{\beta_1, \beta_2, \beta_3} (X, E) \]
Codimension on states of Pandey-Isidrovers
compactification $\mathcal{M}^{P2}_{g,0,\ell,\pi_2}(X,E)$ is smaller
than that of $\mathcal{M}^{BEHV2}_{g,0,\ell_2}(X,E)$.

Algebraic structure of SFT should be related
to $\mathcal{M}^{P2}_{g,0}\cong \mathcal{M}^{BEHV2}$.
Note that a stable finite IR$^d$ action on \( \hat{M} \) is obtained by shrinking the closures of their orbits to a point.

Existence

\( \exists x \quad \hat{M} \)
\[ R \rightarrow \mathcal{A} \]

1 1 \[ f = 1 \]

Closure of calat's containes

1 1 \[ f = 0 \]

\[ 0 - 1 \]

\[ 0 - 0 \]
Strata-wise $\mathbb{R}^1$ action extends to the ambient set $\tilde{\mathcal{V}}_{\mathbb{R}^1}(X, \bar{\sigma})$.

Def: Obstruction bundle data is invariant of strata-wise $\mathbb{R}^1$ action

\[ \exists \mathbb{P} \in \mathcal{P} \times \mathbb{R} \quad \mathcal{O}(\mathbb{P}, \mathbb{P}) \in \mathbb{R} \]

\[ \mathcal{O}_1 \quad x \in \mathbb{R} \cdot \mathbb{P} \]

\[ \mathcal{O}_2 \quad \mathbb{P} \mapsto \mathbb{P} \cdot \mathbb{P} \cdot \mathbb{P} \cdot (X) \]

The isomorphism extends to the closure
If the bundle data is $H^2$ act in 

\[ \Rightarrow \text{If } \Phi \text{ on } M \Rightarrow \text{Hermes becomes } H^2 \text{ in. } \]

\[ \Rightarrow \text{We can choose CF perturbation to be } H^2 \text{ in.} \]

We can use it to prove that when applying Stokes, I component which has nontrivial $H^2$ action does not contribute.
How to construct obstruction bundle data? Which is the invariant?

Two extra properties we need for $E_p(x)$:

1. Compatibility with fiber product description at boundary and corners
2. Invariance with strata-wise $L^2$ action
\( \Theta \) will be a consequence of "component-wise-ness", which I will next describe.

\[
\mathcal{F} = \left\{ (x^1, x^2, \ldots, x^n) \mid x \in \mathbb{R} \right\} \subset \bigwedge (\mathbf{X}, \mathbf{E})
\]

\[
\mathcal{F} = \left\{ [\mathcal{Z}, \mathcal{W}, \mathcal{V}, \mathcal{U}, \mathcal{L}] \mid M_{1,1} \mathcal{F} (x, \mathbf{E}) \right\}
\]

\[
\mathbf{X} = \mathcal{R}_\mathbf{e}(\mathcal{F})
\]
I will define the motion of components. Let

\[ E_p (H) \]

Let \( Z' = \bigcup \mathbb{Z}_a \) decomposition to irreducible component

Then \( Z = \bigcup \mathbb{Z}_a \)

\( Z_a \) may not be irreducible.
\[ K = \prod_{a \in A} x_a \]  
\[ \tau_a = (z_a, \omega, \mu, \lambda, \eta) \]  
\[ \eta = \prod_{a \in A} \eta_a \]  
\[ \eta_a = (z_a, \omega_a, \mu_a, \lambda, \gamma) \]  

Def: Obstruction bundle data is composed wise

\[ \Rightarrow E_{\mathcal{M}}(H) = \bigoplus_{a \in A} E_{\mathcal{M}}(x_a) \]
$E_{\mu_i}(\mathcal{X}_i) \subseteq C^0(\bar{\Omega}^i; \mathbb{R}^{m\times n} \otimes \Lambda_{\mu_i})$
The important condition is

$$E_p(t) = E_{p_1}(x_1) \Theta E_{p_2}(x_2) \Theta E_{p_3}(x_3)$$

Namely, the part of $E_p(t)$ whose support is on $Z_1$ depends only on $x_1, p_1$, and is independent of $x_2, p_2, x_3, p_3$. 

Componentwise $\Rightarrow$ Compatibility with fiber product

$U_1'$ $\quad \xi = 1$

$U_2$ $\quad \xi = 0$

$U_2:1$ $\quad \xi = 1$

$U_1$ $\quad \xi = 0$
\((U_1, U_2) \subset \text{Kuramashi nbd of } p \) 

\( \Rightarrow \quad \exists U_1' + \delta U_2 \subset E_p(U) \) 

\( \Rightarrow \quad \exists U_1' \subset E_{p_1}(U_1') \) 

\( \exists U_2' \subset E_{p_2}(U_2') \) 

\( \Rightarrow \quad U_1' \subset \text{Kuramashi nbd of } p_1 \) 

\( U_2' \subset \text{Kuramashi nbd of } p_2 \)
Thus the issue is to find a
(confirmation)
abstraction between data
who is unafigured wise

Then

\[ \text{Eq. 1} \]
Let us discuss it.

Reference:
A. Prem: arXiv 1809.03409
Fuku: arXive 1808.06106

(I think Ishicawa's proof is basically similar.)

Let me first explain one point which I did not explain in the definition of obstruction bundle data.
$F_p(x) \text{ is smooth wrt } x$

What it mean

**Example**

\( \text{H} = (\overline{z_2}, \overline{z_1}, \overline{w}) \)

\[ u_2 \xrightarrow{\mathcal{F}} \overline{z_2} \quad \overline{z_1} \quad \overline{t} = 1 \]

\[ u, \overline{z_2}, \overline{z_1} \quad \overline{t} = 0 \]

First assume \((\overline{z_2}, \overline{z_1}, \overline{w})\), \((\overline{z_3}, \overline{z_2}, \overline{w})\) is stable
H = (z', z' \cup z, z')  

(z', z' \cup z) \sim (z, z' \cup z) \text{ in DM space.}

\exists \text{ differ} \quad \overline{z}'(\text{thick}) \Delta \overline{z}_1(\text{thick}) \cup \overline{z}_2(\text{thick})
\[ U' \mid \Sigma(t_{\text{thick}}) \text{ may regard as a map} \]

\[ \Sigma_t(t_{\text{thick}}) \xrightarrow{u'_1} X \cap R > N \]

\[ \Sigma_t(t_{\text{thick}}) \xrightarrow{} X \cap R > N \]

\[ E_n(p) \subseteq C^{00}(\Sigma_t(t_{\text{thick}}), U'_1, X) \]

\[ \Theta \subseteq C^{00}(\Sigma_t(t_{\text{thick}}), U'_1, \cap R > N) \]
We require

1. \( F_{\mu}(b) \) depends only on \( \mathbb{H}'(\text{thick}) \)
   and its elements are supported on \( \mathbb{H}'(\text{thick}) \).

2. \[ \forall \mathbb{H}'(\text{thick}) \rightarrow E_{\mu}(\mathbb{H}) \text{ is smooth.} \]
$\exists \ e_1, \ -\ , \ e_r$

$e_i(w) \in \text{ a map}$

$\mathbb{L}^2 (\mathbb{R}^+(\text{thick}) \times \mathbb{X}) \xrightarrow{\mathcal{A}} \mathbb{L}^2 (\mathbb{R}^+(\text{thick}) \times \mathbb{Y} \times \mathbb{V})$

$s_2 \ \forall \ \mu \in \mathbb{M} \Rightarrow \ A \in C^m$

$\exists \ e_i(w), \ -\ , \ e_k(w) \text{ is a basis of } E_H(\mathbb{R})$
When \((\bar{z}, \bar{z}_W \bar{w})\) is not stable.

\[ H^a (\bar{z}_a, \bar{z}_a W w_a) \text{ is not stable,} \]

\[ \text{if } \forall a : \bar{z}_a \rightarrow X \quad \int u^* w > 0 \]

\[ \text{or} \quad \forall a : \bar{z}_a \rightarrow N \times N \quad \int u^* (\Delta) > 0 \]
Take $\mathbb{Z}_a < \mathbb{Z}_a$ to

1. $(\mathbb{Z}_a, \mathbb{Z}_a, u \mathbb{Z}_a, Z_a)$ stable
2. $\mathbb{Z}_a$ is an immersion at $p_5 \in \mathbb{Z}_a$
3. $\mathbb{Z}_a : \mathbb{Z}_a \to N$ is an immersion at $p_5 \in \mathbb{Z}_a$

Each $\mathbb{Z}_a$: take $W_{ai} C X \in N$
$(w', w', z', w, z) \in B_\varepsilon (y)$

take $z_{a,i}^+$ s.t. $h'(z_{a,i}^+) \in W_{a,i}$

and $(w', w', z', w, z') \in (z, \bar{w}, z, \bar{w}, z')$
in DM modular.

We then reduce the situation to the stiff case.

Smoothness of $E_p(x)$ with $x$ is real.

To show $\mathcal{D}_{\mathcal{X}}(C-E_p(x))$ is an orbifold (gluing analysis. See FOO00 arXiv: 1603.02020)
Last step

How to construct obstruction bundle
data which is component wise

Recall

Baby example

Hilbert bundle
over a Hilbert
manifold
+ Fredholm section
2 = \delta'(0) \text{ compact}

\forall \varepsilon > 0 \text{ there } E_\varepsilon \subset E_\varepsilon' \text{ finite dimension}

\text{st } E_\varepsilon \cap \lim D_\varepsilon = \varepsilon.

\text{Take still } U_\varepsilon \text{ and ext } E_\varepsilon \text{ st }

\exists G \subset U_\varepsilon \Rightarrow E_\varepsilon(G) \cap \lim D_\varepsilon = \varepsilon.

\forall \varepsilon : \varepsilon H \subset C U_\varepsilon \text{ compact mult}
\[ \text{Core } \mathcal{Z} = \bigcup_{1 \in \mathcal{Z}} \text{Int } K_{1} \]

\[ p \in \mathcal{Z} \quad \text{L.H.S of } \mathcal{Y} \]

\[ F_{p}(t) = \bigcup_{\gamma \in \mathcal{K}_{p}} E_{p}(\gamma) \]

"Imitate it. But work a bit harder to keep component-wise mean"
\( M(X, \mathcal{E}) \)

smallest \( T \)

\[ \exists m(x, E) = \emptyset \]

\[ p = (z_T, u_0) \cap M(X, \mathcal{E}) \]

\( T \)

\[ T_0 \subset C^0(\Sigma_T, u^{x+T_X, \Theta}) \]

\[ \lim B_{H^1} + E_0^0 = C^0(\cdot) \]

Choose \( E_0^0 \subset C^0(\Sigma_T, u^{x+T_X, \Theta \mathcal{V}^o}) \)
Let \( X = (Z', u') \) be a chart of \( \mathbb{R}^n \) such that \( u' \) is an embedding.
$\mathcal{E}_0 \subseteq C^0(\Omega, \mathbb{R}^n \times \mathbb{R}^{n'})$

$\mathcal{E}_0(x) \subseteq C^0(\Omega, \mathbb{R}^n \times \mathbb{R}^{n'})$

$t_{\mathcal{E}} \times \mathbb{R} \subseteq T_{x_{\mathcal{E}}} \times \mathbb{R}$

$D_{\mathcal{E}_0} \subset \mathcal{E}_0(x) = C^0(\mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{n'})$

$\cup (C_0^2 \times \mathbb{R}) \subset \mathbb{R}_0^+ \times \mathbb{R}$
The next is the same as the last case.

Inductive step

\[ p_n \]

\[ p \]
\[ \hat{\mathbf{p}} = \hat{\mathbf{p}}_1 + \hat{\mathbf{p}}_2 \quad \hat{\mathbf{p}}_i \in M(E_i) \]

\[ M(E_i) \subseteq \text{null of } \mathbf{p}_{i,i}, \quad i = 1, \ldots, J_i \]

\[ \mathbf{f}_i \sim \mathbf{p}_{i,i} \quad \mathbf{E}(\mathbf{p}_{i,i}) \text{ is given} \]

\[ \mathbf{f}: \Omega_{\mathbf{p}_{i,i}} \rightarrow \mathbb{R} \]
We can define \( E_p(n) \).

In this way we define \( E_p(n) \) if \( n \) is close to \( 2N \).

Then extend to \( \hat{M} \).