Zoology in the Hénon Family: Renormalization nearby chains of homoclinic tangencies

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Projet ERC 818737 « Emergence of wild differentiable dynamics »
1) Definition of homoclinic tangencies, chains of homoclinic tangencies and their unfolding, statement of some consequences of our results.

2) Explanation of the techniques in the context of quadratic-like maps of Banach Algebra.

3) Explanation of the techniques in the context of Hénon-like maps of Banach Algebra.
Why studying dynamics nearby homoclinic tangencies?

A) These form an open set of dynamical systems in many categories:

1970 Newhouse, for surface, smooth dynamics,
1997 Buzzard for surface automorphism of $\mathbb{C}^2$
1996 Bonatti-Diaz: for smooth dynamics of any dimension >2
2017 Biebler Cs-Blender for degree 2 polynomial automorphism of $\mathbb{C}^3$

B) Conjecturally these form the interior of the complement of uniformly hyperbolic dynamics (A conjecture of Palis 90’s)

C) The dynamics nearby homoclinic tangency is very rich (Universal dynamics: Gochenko-Shinikov-Turaev, Bonatti, Turaev).
Aim of this mini-course:

« Introduce » a formalism for renormalization nearby chain of heteroclitic tangencies to deduce several phenomena:

(i) Renormalization lamination in the space of Hénon like maps, with long leaves of codimension 1 from Jacobean 0 to 1.

(ii) Renormalization lamination in the space of Hénon like maps, with leaves of codimension 2 nearby Jacobean b=1.

(iii) Renormalization of unfolding of homoclinic tangencies as Hénon-like maps for the first time without lost of derivatives.

(iv) Milnor swallow nearby twice non-degenerate homoclinic unfolding and the coexistence phenomena which follow. Sharp description of part of the coexistence locus (laminations etc).

(v) Renormalization of unfolding of any chain of heteroclinic tangencies as composed Hénon-like families (without lost of derivatives). Coexistence phenomena which follow. Sharp description of some coexistence locus (laminations etc).

(vi) Existence of wandering Fatou component for polynomial automorphisms (with Biebler).
Lyapunov exponent of the point 0 under the Hénon map $h_{a,b}(x, y) = (x^2 + a - by, x)$ depending on the parameter $(a, b)$. 
Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $D_{\mathbb{K}}(R)$ be the segment or disk of radius $R$.


For all $R > 0$, $\delta > 0$, $r \geq 1$, there exists a domain $\Delta$ of parameters $(a, b)$ of the Hénon family $(h_{a,b})_{a,b}$, there exists $n \geq 1$, such that for every $p = (a, b) \in \Delta$, there exist a domain $D_p \subset \mathbb{K}^2$, and an embedding $\phi_p \in C^r(D_p, \mathbb{K}^2)$ and a diffeomorphism $\psi \in C^r(\Delta, D_{\mathbb{K}}(R)^2)$ such that:

$Rh_p := \phi_p^{-1} \circ h_p^n \circ \phi_p$ is defined on $D_{\mathbb{K}}(R)^2$ and $\delta - C^r$ close to $(x, y) \mapsto ((x^2 + c_2 + y)^2 + c_1, 0)$, with $(c_1, c_2) = \psi(p)$.

Lyapunov exponent of the point 0 under the Hénon map $h_{a,b}(x, y) = (x^2 + a - y, bx)$

Details of the previous figure

Restriction of the previous figure

Lyapunov exponent of the point 0 and $a$ under the maps $S_{a,b}(x) = (x^2 + a)^2 + b$. 
Lyapunov exponent of the point 0 under the Hénon map
\( h_{a,b}(x, y) = (x^2 + a - y, bx) \)

Details of the previous figure

Restriction of the previous figure

Lyapunov exponent of the point 0 and \( a \) under the maps
\( S_{a,b}(x) = (x^2 + a)^2 + b \)

This implies that the hyperbolic continuation of the attracting cycles of a quadratic mapping cannot fully explain the coexistence of infinitely many sinks of any Hénon maps (a negative answer to a question by Van Strien).

This implies also many coexistence phenomena.
What is new since 2017:

We deal with Banach algebra in order to deal with:

\(a\) Renormalization of families parametrized by a Banach space (and obtain renormalization operator).

\(b\) Be able to apply our result for homoclinic bifurcation in a partially hyperbolic setting (with unstable direction equal to 1).

\(c\) The theorems are easier to apply (no need to check that the definition domain of the compositions).

\(d\) The computations are softer.

\(e\) The case \(K = \mathbb{C}\) is addressed.
II) Misiurewicz Renormalization of (composed) quadratic-like maps

For unimodal maps, a quadratic homoclinic tangency is a critical point which is sent to a repulsive periodic point.

**Quadratic like map**

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $\mathbb{D}_{\mathbb{K}}(R) = \{ x \in \mathbb{K} : |x| < R \}$.

**Definition**

For $r \geq 2$ and $\delta > 0$, a map $Q : \mathbb{D}_{\mathbb{K}}(R) \to \mathbb{K}$ is a $\delta - C^r$ quadratic-like if:

$$Q(x) = x^2 + a + \eta(x) \in \mathbb{K} \text{ for a } \delta - C^r \text{ small } \eta.$$ 

It is normalized if $\eta(0) = D\eta(0) = D^2\eta(0) = 0$.

**Definition**

A (pre)-renormalization of $Q$ of period $n \geq 2$ is the data of an affine map $\phi : \mathbb{K} \to \mathbb{K}$ such that:

$$\mathcal{R}Q := \phi^{-1} \circ Q^n \circ \phi$$

is a normalized quadratic like map well defined on $\mathbb{D}_{\mathbb{K}}(R)$.

**Problem**

How to define the operator $\mathcal{R}$ with nice hyperbolic behavior?
How to obtain nice $C^r$ bounds without too much computations?
Let $\mathbb{A}$ be a **commutative, unital, Banach algebra**: $\|x \cdot y\| \leq \|x\| \cdot \|y\| \ \forall x, y \in \mathbb{A}$.

**Examples**

- Let $d \geq 1$ and $\mathbb{A} := \mathbb{K}[X]/(X^{d+1})$ endowed with $\|\sum_{i=0}^{d} a_i \cdot X^i\| = \sum_{k=0}^{d} \frac{k!}{(k-d)!} |a_k|$ is a Banach algebra.

- If $\mathcal{P}$ is an open set of a Banach space, the space $\mathbb{A} := C^d(\mathcal{P}, \mathbb{K})$ is a Banach algebra endowed with the norm:

\[
\| (x_p)_{p \in \mathcal{P}} \|_{C^d} = \| (x_p)_{p \in \mathcal{P}} \|_{C^{d-1}} + \| (\partial^k_p x_p)_{p \in \mathcal{P}} \|_{C^{d-1}} = \sum_{k=0}^{d} C_d^k \| (\partial^k_p x_p)_{p \in \mathcal{P}} \|_{C^0}.
\]

**Definition**

Any cartesian product $\mathbb{A}^n$ is endowed with its canonical structure of $\mathbb{A}$ module: $x \cdot (x_i)_{1 \leq i \leq n} = (x \cdot x_i)_{1 \leq i \leq n}$.

A linear map $g$ from $\mathbb{A}^n$ into $\mathbb{A}^m$ is $\mathbb{A}$—**linear** if it is a continuous linear map such that $g(x \cdot y) = x \cdot g(y)$ for all $x \in \mathbb{A}, y \in \mathbb{A}^n$. **Note that if** $n = 1 = m$, **then** $g = x \mapsto a \cdot x$ **for** $a \in \mathbb{A}$.

**Example:**

If $f : \mathbb{K} \mapsto \mathbb{K}$ is of class $C^3$ and $\mathbb{A} := \mathbb{K}[X]/(X^3)$, then the following is continuous:

\[
F : \sum_{i=0}^{2} a_i X^i \in \mathbb{A} \mapsto J_0^2 f(a_0 + a_1 X + a_2 X^2) = f(a_0) + D_{a_0} f \cdot a_1 \cdot X + (D_{a_0} f \cdot a_2 + \frac{1}{2} D_{a_0}^2 f \cdot a_1^2) \cdot X^2 \in \mathbb{A}
\]

And its derivative at every point $a := \sum_{i=0}^{2} a_i X^i$ is the $\mathbb{A}$—linear map:

\[
D_a F : b = \sum_{i=0}^{2} b_i X^i \mapsto D_{a_0} f \cdot b_0 + (D_{a_0} f \cdot a_1 \cdot b_0 + D_{a_0} f \cdot b_1) \cdot X + (D_{a_0} f \cdot b_2 + (D_{a_0} f \cdot a_2 + \frac{1}{2} D_{a_0}^2 f \cdot a_1^2) \cdot b_0 + D_{a_0}^2 f \cdot a_1 \cdot b_1) \cdot X^2
\]

and so $D_a F : b \mapsto D_a F \cdot b$, with $D_a F = D_{a_0} f + D_{a_0}^2 f \cdot a_1 \cdot X + (D_{a_0}^2 f \cdot a_2 + \frac{1}{2} D_{a_0}^3 f \cdot a_1^2) \cdot X^2 \in \mathbb{A}$.
Definition
A subset of a Banach space is regular if it is the closure of its interior. A \( C^1 \) map \( f \) from a regular set \( R \subseteq \mathbb{A}^n \) into \( \mathbb{A}^m \) is of class \( C^1_{\mathbb{A}} \) if its differential \( D_y f \) is \( \mathbb{A} \)-linear at every point \( y \in R \). The map \( f \) is of class \( C^{r+1}_{\mathbb{A}} \) if its differential \( Df \) is of class \( C^r_{\mathbb{A}} \).

Example (Functor \( C^d \)-Jet at 0):
Let \( J^d_0 \mathbb{K} := \mathbb{K}[X]/(X^{d+1}) \), \( d \geq 1 \). For any \( r \geq 0 \), a \( C^d \)-map \( f : \mathbb{K} \mapsto \mathbb{K} \) is of class \( C^{r+d} \) iff the following is of class \( C^r \):

\[
J^d f : \sum_{i=0}^d a_i X^i \mapsto \sum_{i=0}^d \partial_x^i \left[ f(a_0 + a_1 x + \ldots + a_d x^d) \right]_{x=0} \frac{X^i}{i!}
\]

In this example, we can substitute \( \mathbb{K} \) to any Banach algebra \( \mathbb{A} \) to obtain:

Proposition
Let \( J^d_0 \mathbb{A} := \mathbb{A}[X]/(X^{d+1}) \), \( d \geq 1 \). A map \( g \) is of class \( C^{d+r}(\mathbb{A}^n, \mathbb{A}^m) \) iff \( J^d g \) is \( C^r_{\mathbb{A}} \). This functor \( J^d_0 \) is bi-Lipschitz on its range.

Proposition
If a map \( f : \mathbb{A}^n \to \mathbb{A}^m \) is of class \( C^r_{\mathbb{A}} \), then \( D_y f \) is \( \mathbb{A} \)-multilinear: \( D_y f(x_1, y_1, \ldots, x_r, y_r) = x_1 \cdots x_r \cdot D_y f(y_1, \ldots, y_r) \).

Definition
A family \( (g_p)_p \) of maps \( g_p \in C^{d+r}(\mathbb{K}^n, \mathbb{K}^m) \) is of class \( C^{d,d+r} \) if \( \partial_x^i \partial_p^j f \) exists continuously for every \( i + j \leq r + d, j \leq d \).

Proposition
A family \( (g_p)_p \) of maps \( g_p \in C^{d+r}(\mathbb{K}^n, \mathbb{K}^m) \) is of class \( C^{r,d+r} \) iff \( f : (y_p)_p \in \mathbb{A}^n \mapsto (g_p(y_p))_p \in \mathbb{A}^m \) is of class \( C^r_{\mathbb{A}} \).
Moreover the functor is bi-Lipschitz on its range.
Let $K = \mathbb{R}$ or $\mathbb{C}$. Let $D_A(R) := \{x \in A : \|x\| < R\}$.

**Definition**
For $r \geq 2$ and $\delta > 0$, a map $Q : D_A(R) \to A$ is a $\delta - C^r_A$ quadratic-like if:

$$Q(x) = x^2 + a + \eta(x) \in A$$

for $a \in A$ and a $\delta - C^r_A$ small $\eta$.

It is **normalized** if $\eta(0) = D\eta(0) = D^2\eta(0) = 0$.

**Example**
Let $V \subset K$, $\mathcal{P} := V \times P$, with $P := \{\eta \in C^r_K(D_K(R), K) : \eta(0) = D\eta(0) = D^2\eta(0) = 0, \|\eta\|_{C^r} < \delta\}$ and $A := C^r(\mathcal{P}, K)$. These are normalized $\delta - C^r_A$ quadratic-like maps:

$$Q : (x_p)_{p \in \mathcal{P}} \in D_A(R) \mapsto (x_p^2 + \nu + \eta(x_p))_{p = (\nu, \eta) \in \mathcal{P}} \in A$$

$$RQ : (x_p)_{p \in \mathcal{P}} \in D_A(R) \mapsto (x_p^2 + a_p + R_p(x_p))_{p \in \mathcal{P}} \in A,$$

given any $(a_p)_p \in A$ and any $C^{r,r'+r}$- small family $(R_p)_p$ of $R_p \in D_p(\delta)$.

**Aim:**
Define $R : p \in \mathcal{P} \mapsto (a_p, R_p)_p \in K \times P$ such that $RQ$ is a renormalization of $Q^n$.
Let $B^e \subset A$ be a regular, convex subset of $A$ of diameter $\leq 1$. Let $c < 1$ and $r \geq 2$.

**Definition**

A $C^r_A$-hyperbolic map $F$ is a diffeomorphism from $B^F \subset B^e$ onto $B^e$ such that $\|DF^{-1}\|_{C^0} < c$.

The $C^r$-distortion of $F$ is $N_r(F) := \|\mathcal{X}\|_{C^r} + \|D\mathcal{X}^{-1} \cdot D^2\mathcal{X}\|_{C^{r-2}}$, with $\mathcal{X}$ the inverse of $F$.

The fixed point of $F$ is denoted by $W^s(F)$.

**Proposition (Bounds)**

For every $K_0 > 0$ and $r \geq 2$, there exists $K_1 > 0$ such that for every sequence of hyperbolic maps $(F_i)_{i \in \mathbb{N}}$ with $C^r$ distortion bounded by $K_0$, it holds:

(i) the map $F_n \circ \cdots \circ F_1$ is hyperbolic, for every $n \geq 1$.

(ii) its $C^r$ distortion of $F_n \circ \cdots \circ F_1$ is bounded by $K_1$, for every $n \geq 1$.

**Definition**

A $C^r_A$-folding map $G$ is a map from $B^G \subset B^e$ onto $B^e$ which has a unique critical point $\xi$, which is furthermore in the interior of $B^G$ and non-degenerate: its quadratic term satisfies $q := \frac{1}{2} D^2G(\xi) \in U(A)$, with $U(A)$ the set of invertible elements.

The geometry of $G$ is $C^r$-bounded by $R > 0$ if $\|G\|_{C^r} < R$, $d(\xi, \partial B^G) > 1/R$, $\|q^{-1}\| < R$. 

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We recall that: \( \mathbb{D}_A(R) = \{ x \in A : \|x\| < R \} \).

**Theorem**
For any \( r \geq 3, \ R \geq \text{diam}B^e \) and \( \delta > 0 \), for any \( N \geq 0 \) sufficiently large, for any hyperbolic map \( F \) and any folding map \( G \) with \( C^r \)-distortion bounded and \( C^r_{\mathbb{A}} \)-geometry bounded by \( R \), the following property holds true:

There is an affine map \( \psi \) such that \( \mathcal{R}Q = \psi^{-1} \circ G \circ F^N \circ \psi \) is a normalized \( C^r_{\mathbb{A}} \)-\( \delta \)-quadratic-like map defined on \( \mathbb{D}_A(R) \). Also \( x_0 := \psi(0) \) is \( \delta \)-close to \( W^s(F) \) while \( F^N(x_0) = \xi \) is the critical point of \( G \). Moreover, with \( \lambda := DF^N(x_0) \) and \( a := \mathcal{R}Q(0) \) is equal to \( q \cdot \lambda^2 \cdot (G(\xi) - x_0) \).

**Proof:**
Using the functor \( J^{r-3} \) it suffices to show the case \( r = 3 \). Let \( \psi(x) = x_0 + q^{-1} \lambda^{-2} \cdot x \) and \( \phi(x) = \xi + q^{-1} \lambda^{-1} \cdot x \). Using the distortion bound, \( \phi^{-1} \circ F^N \circ \psi \) is \( C^3 \)-close to the identity, while \( \psi^{-1} \circ G \circ \phi \) is a normalized quadratic like map with constant term \( q \cdot \lambda^2 \cdot (G(\xi) - x_0) \).

**Example**
Using this result with \( \mathbb{A} = C^1(\mathbb{D}_c(1), \mathbb{C}) \), this implies that there are baby Mandelbrot sets nearby any strictly post-critically finite parameter of the quadratic family.

Using this result with \( \mathbb{A} := C^r(\mathcal{P}, \mathbb{C}) \) and \( \mathcal{P} := V \times \mathbb{D}_p(2\delta) \), and with \( Q : (x_p)_p \mapsto (x_p^2 + \nu + \eta(x_p))_{p=\nu,\eta} \), the above theorem defines an analytic map \( \mathcal{R} : p \in \mathcal{P} \mapsto (a_p, \eta_p) \in \mathbb{C} \times \mathbb{D}_p(\delta) \) such that \( (\partial_p a_p)^{-1} \simeq \lambda^{-2} \) is small. Also the Carathéodory metric of \( \partial_p \eta \) is smaller than 1. Hence \( \mathcal{R} \) is a hyperbolic map. Its 1-codim. stable manifold is formed by infinitely renormalizable quadratic like maps.
Let $\mathcal{J} := \mathbb{Z}/J\mathbb{Z}$ for $J \geq 1$ or $\mathcal{J} := \mathbb{N}$.

**Theorem**

For any $r \geq 3$, $R \geq \text{diam} B^e$ and $\delta > 0$, for any $(N_j)_{j \in \mathcal{J}}$ sufficiently uniformly large, for any families $(F_j)_{j \in \mathcal{J}}$ and $(G_j)_{j \in \mathcal{J}}$ of hyperbolic and folding maps $F_j$ and $G_j$ with $C^r_A$-distortion and $C^r_A$-geometry bounded by $R$, denoting $\xi_j$ and $q_j$ the critical point and quadratic term of $G_j$ and also $\lambda_j := D F_j^{N_j}(x_j)$ with $x_j = F_j^{-N_j}(\xi_j)$, if there is $(\gamma_j)_{j \in \mathcal{J}} \in U(\mathbb{A})^J$ small satisfying:

\[
(\star) \quad \gamma_{j+1} = q_j \cdot \lambda_j^2 \cdot \gamma_j^2,
\]
then there exist affine maps $(\psi_j)_{j \in \mathcal{J}}$ such that $Q_j = \psi_j^{-1} \circ G_j \circ F_j^{N_j} \circ \psi_j$ is a normalized $C^r_A$-$\delta$-quadratic-like map defined on $\mathbb{D}_A(R)$. Moreover, $\mathcal{K}Q_j(0)$ is equal to $a_j := \gamma_{j+1}^{-1} \cdot (G_j(\xi_j) - x_{j+1})$.

**Proof:**

Again it suffices to prove the case $r = 3$. Let $\psi_j(x) = x_j + \gamma_j \cdot x$ and $\phi_j(x) = \xi_j + \lambda_j \cdot \gamma_j \cdot x$. Using the distortion bound, $\phi_j^{-1} \circ F_j^N \circ \psi_j$ is $C^3$—close to the identity, while $\psi_j^{-1} \circ G_j \circ \phi_j$ is a quadratic-like map with constant term $a_j$. \qed

**Examples**

- If $\mathcal{J}$ is finite, this theorem enables to conjugate the composition $\bigcup \mathcal{J} \cdot G_j \circ F_j^{N_j}$ to $\bigcup \mathcal{J} \cdot Q_j$. One can deduce for instance the density of a lamination of co-dimension $d$ of infinitely renormalizable maps in the total bifurcation locus of rational functions of degree $d$. Indeed $(\star)$ is always satisfied if $U(\mathbb{A}) = \exp(\mathbb{A}) \cdot \{x \in \mathbb{A} : x^2 = 1\}$, as for $\mathbb{A} = \mathbb{R}$ or $\mathbb{C}$. Indeed with $a_j \cdot \lambda_j^2 = \delta_j \exp \ell_j$ we take:

\[
\gamma_j = \delta_{j+1} \cdot \exp \sum_{k \geq 0} 2^{-k-1} \cdot \ell_{j+k}.
\]

- If $\mathcal{J}$ is infinite, this might be useful to construct examples of wandering Fatou component for some entire maps of $\mathbb{C}$.  

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