Abstract approaches to regularizing moduli spaces of pseudoholomorphic curves.

**Approach #1**: "Euler class on Banach orbifolds" [Siebert]

* Gromov-Witten invariant

* Kuranishi section for
  - nontrivial Deligne-Mumford space
  - neighborhood of nodal curve

**Approach #2**: finite dimensional reductions

Fukaya-Ono-Oh-Ohba 1999

Joyce

McDuff-W.  Kuranishi atlas

Kuranishi structure (of germs)  \(\rightarrow\) d-orbifold
Theorem 0.1 Let \((M, \omega)\) be a closed symplectic manifold with a tame almost complex structure \(J\). Then the space \(C(M; p)\) of stable parametrized marked complex curves in \(M\) of Sobolev class \(L^p_\omega\) (Definition 3.1) is a Banach orbifold. Moreover, there is a Banach orbibundle \(E\) over \(C(M; p)\) with fiber \(L^p(C; \varphi^*T_M \otimes \Omega_C)\) at \((C, x = (x_1, \ldots, x_k), \varphi : C \to M)\) with an oriented Kurumishi section \(s\) (Definition 1.15) with \(\delta(C, x, \varphi) = \partial_1 \varphi\cdot \). The zero locus of \(s\) is the set \(C^{\text{orb}}(M, J)\) of stable pseudo holomorphic curves in \((M, J)\) (Definition 3.5), which is a locally finite dimensional Hausdorff space with compact components.

Let \(M_{g,k}\) be the moduli space of Deligne-Mumford stable \(k\)-marked algebraic curves of genus \(g\) with the convention \(M_{g,k} = \{pt\}\) whenever \(2g + k \leq 3\). The localized Euler class \(GW_{g,k}^{M,J} \in H^*(C^{\text{orb}}(M, J))\) associated to \((E, s)\) (Theorem 1.21) gives rise to GW-correspondences (Definition 7.2)

\[
GW_{g,k}^{M,J} : H^*(M)^{\otimes k} \to H^*(M_{g,k}),
\]

that are invariants of the symplectic deformation type of \((M, \omega)\). They coincide with the ones defined in [RuTi12] in case \((M, \omega)\) is semi-positive. (by geometric regularization - e.g. [McDuff-Salamon]

**Basic example:** \(\overline{\mathcal{M}} = \text{Euler}(\mathbf{w} \mathbf{v}; \mathbf{w}_0, \mathbf{v}_0) \in H^{*}(\overline{\mathcal{M}} = \mathbf{s}'(\mathbf{0}); \mathbb{Q})\)

\[
\overline{\mathcal{M}} = \{u \in W^{1,1}(\mathbf{P}, M) \mid \tilde{\sigma}_u = 0, u = 0 \text{ at } \mathbf{P}\} / \text{Aut}(\mathbf{P}, i, \infty) \xrightarrow{\text{ev}} M
\]

\(\xrightarrow{\alpha} GW_{0,1}^{M,J} : H^{*}(M) \to \mathbb{Q}\)

\[
\alpha \mapsto \langle \text{ev}^{*}\alpha, [\overline{\mathcal{M}}] \rangle = \langle \alpha, \text{ev}_{\mathbf{0}}(\overline{\mathcal{M}}) \rangle
\]

\(H^{*}(\overline{\mathcal{M}}) \to H^{*}(M)\)

**Rmk:** semi-positive GW is constructed the same way with

- ideally \([\overline{\mathcal{M}}] = \text{classical fundamental class from } \overline{\mathcal{M}} = \overline{\mathcal{O}}(\mathbf{0}) \text{ at the marked curve}\)

- for regular \(J\) "being" a compact manifold

- generally, \(\cdots\) give "pseudocycle"

Then Euler class axiom 0 says \(\text{Euler}(s) = \mathbf{s}'(\mathbf{0}) \simeq \overline{\mathcal{M}}\) since \(H^{*}(\overline{\mathcal{O}}(E_{M, p})\)

\(\overline{\mathcal{O}}\)
Guiding Questions for studying regularization approaches via abstract perturbations/virtual fundamental class
\[ \overline{M} = s'(0) \oplus \left[ (s-r)'(0) \right] | \text{Euler}(s) = [\overline{\mathcal{H}}] \]

what is the abstract form of Section 5?

- topological \( E \)
- Banach bundle \( S \)
- \( R^b \times V \) \( \Theta \) \( R,5 \)
- \( E_0 \), \( s_0 \) continuous (wrt \( \mathbb{R} \))
- family of Fredholm sections \( s_0 \in \mathcal{C}(V, E_0) \)
- differentiable relative \( \mathbb{R}^b \)

how is \( S \) constructed for pseudoholomorphic curve moduli spaces?

from local Fredholm descriptions:

1. near smooth curves as before, where nondifferentiability of reparameterization is no issue since transition maps are only required to be continuous

\[ \{v(0) \in H_v, \{v(i) \in H_v_i\} = S \rightarrow B = W^{1,2}(M_\text{red}, H) \text{ absor} \rightarrow S' = \{v(0) \in H_v', \{v(i) \in H_v'i\} \}

\rightarrow \text{continuous transition map } v \rightarrow v_{\text{op}}. \]

2. for \( M_{g,n} \oplus pt \)

3. for nodal curves
Analytic issues appear in construction of Kuranishi structure

\[ \mathcal{F} \xrightarrow{\tau} \mathcal{E} \]

since \( \mathcal{T} \) is required to be "differentiable relative \( \mathbb{R}^k \)" in local differentiable structures.

\[ \mathcal{B}' \xleftarrow{v} \mathcal{B} = W^0(\mathcal{M}_{\mathbb{A}^k}) \]

This requires construction of stabilizations in local slices that "transform as if the transition map was differentiable":

\[ \mathcal{F}' = \Phi^* \mathcal{F} \] should be a differentiable finite rank bundle.

i.e. both \( \mathcal{T} \) and \( \tau \rightarrow \widehat{\Phi} \) should be differentiable.

although \( \widehat{\Phi} \) covers nondifferentiable map \( \Phi : \mathcal{S} \rightarrow \mathcal{S}' \).

(This can be achieved by "geometric construction" of \( \mathcal{F} \) - from \( \mathcal{S}'^{0,1} \) on "universal curve" - which can be reinterpreted as generalized perturbation of \( \mathcal{J} \).)
2. Local Fredholm description for $\overline{M}_{g,k} \times \text{pt}$, "Deligne-Mumford space"

Ex: $\overline{M}_{0,1} = \{ \text{pt} \} = \{(P^1, i, 0)\}_{(0,1)}^{(0,1,\infty)}$ "fixed marked points"

general definition for fixed (closed, oriented) surface of genus $g$

$M_{g,k} = \{(\Sigma, j, (z_1, \ldots, z_k)) \mid j \text{ complex structure}, z_1, \ldots, z_k \in \Sigma \text{ distinct}\} \bigcup (\Sigma, \psi_j, (\varphi(z_1), \ldots, \varphi(z_k))) \forall \psi \in \text{diffeomorphism}

$\overline{M}_{g,k} = M_{g,k} \cup \{ \text{nodal Riemann surfaces arising from degeneration of } j \text{ or coincidence of marked points} \}$

Ex: For $g=0, k=1$ can use uniformization theorem to fix representatives

$\Rightarrow M_{0,1} \simeq \{(P^1, i, (0, 1, \infty)) \mid z \in P^1 \setminus \{0, 1, \infty\}\} \ast \text{ with } z_1 = 0, z_2 = 1, z_3 = \infty \ast \text{ no remaining cv. relation}$

$\overline{M}_{0,1} = M_{0,1} \cup \{\bigcirc, \bigcirc, \bigcirc\}$

"compactification for $z \to \infty$"

Fredholm description for $\overline{\partial}_j$ on $C(M; p) = \{(\Sigma, j, u : \Sigma \to M, z) \mid j \text{ cr. str. }\}$

$\overline{\partial}_j : (\Sigma, j, u, z) \mapsto \overline{\partial}_{j,z} u = \frac{1}{2}(du + J du \circ \varphi_j)$

"locally cr. str. or for $\Sigma = P^1$"

$(\Sigma, j, u, \varphi_j, z) \mapsto \overline{\partial}_j (u \circ \varphi_j)$

$\Delta \text{ differentiability fails }$ for $\overline{M}_{g,k} \times W^p \to L^p$

"differentiability relative $\overline{M}_{g,k}$"

$C(M; p) \xrightarrow{\text{locally}} M_{g,k} \times W^p(\Sigma, M) \xrightarrow{\overline{\partial}_j} E = L^p(0,1)$-forms

$\forall j, \langle j, u \rangle \mapsto \overline{\partial}_j u = \overline{\partial}_{j,z} u$

Each $\overline{s}_j$ is $C'$, Fredholm $\overline{s}_j$ is continuous w.r.t. $j$
3. local Fredholm description near nodal curves

\[ \begin{align*}
\{ v(0) \in H_0, v(1) \in H_1 \} &= \mathcal{F} \\
&\to \text{BunB modal} \\
&\to (L \times V)(v, v, v) \\
\end{align*} \]

\( \Delta \) pregluing \( (R, v_-, v_+) \to v_+ \#_R v_- \) is not injective

Kernel of pregluing

\[ \text{algebraic geometry} \]

\[ \text{Siebert} \]

\[ \text{polyfold theory} \to \text{pregluing is a local chart of polyfold BunB modal} \]
"gluing after stabilization"

\[ F_\pm \xrightarrow{\mathcal{H}_\pm} E_\pm \]

\[ \mathcal{B} \supseteq E_\pm(0) \]

local stabilizations near \( V, V_+ \)

(for simplicity of forgetting to encode (matching at node \( V_0 = V_+(m) \))

\[ \tilde{F} \downarrow \quad \tilde{s} \]

natural transition map

\[ \tilde{U} \supset \tilde{S}^{-1}(0) \leftarrow \tilde{U} \leftarrow \tilde{D} \times U_+ \times U_+ \]

\[ \begin{cases} (f, f) \\ (f, f) \end{cases} \]

extended transition map

constructed by stabilized gluing s.t. \( f = 0 \)

via Newton iteration & choice of \( \varepsilon = \varepsilon + \gamma = 0 \)

extended transition map doesn't naturally arise from an ambient space of "not reachable curves"
#2 Regularization via finite dimensional reductions

**Executive summary:**

\[ \mathcal{M} = \{ s^{i}(0) \} \]
\[ \downarrow \text{as} \]
\[ \bigcup \text{category bundle} \]
\[ \downarrow \text{finite dimensional reductions} \]
\[ \subseteq \text{Mor} \]

**Regularization theorem:**

\[ \exists \mathcal{P} \subset \{ \text{sections } \nu : U \to \mathcal{E} \} : \]

\[ \forall \mathcal{P} \subset \mathcal{P} : [ (s+\nu)^{-1}(0) ] \text{ compact manifold} \]
\[ \forall \nu, \tau \in \mathcal{P} \exists \text{ cobordism } [ (s+\nu)^{-1}(0) \sim (s+\tau)^{-1}(0) ] \]

\[ \Rightarrow [ \mathcal{M} ] = [ [ (s+\nu)^{-1}(0) ] ] \]

** Remark:** This trades some analytic issues in approach #1 (more oo dimensional) for topological issues.

Eg. can make compatible perturbations \( s_i + \nu : \text{Id} \)

so that \( \bigcup (s+\nu)^{-1}(0) \) is locally homeomorphic to \( \mathbb{R}^n \)

but need to ensure Hausdorff & compactness property - which is nontrivial for quotient topologies.