Polyfold - Fredholm theory

M-polyfold bundles and Fredholm sections

Literature:
- Hofer-Wysocki-Behnke
- Hofer - surveys
- Feliu-Fish-Golovko-Wehrheim: "Polyfolds - A first and second look"
\[ \text{Ex}: f: \mathbb{R}^n \to \mathbb{R} \text{ Morse function, let } f = 0 \]

\[ \mathcal{M}_{\text{Morse}}(\mathbb{R}^n, \mathbb{R}^n) = \mathcal{E}^{-1}(0) \quad \text{for } M\text{-polyfold Fredholm section } \mathcal{E} \]

\[ |\mathcal{E}| = \bigcup_{L > 0} H^1([-L,L], \mathbb{R}^n) \cup H^1((0,\infty), \mathbb{R}^n) \times H^1((0,0], \mathbb{R}^n) \]

\[ \text{second countable metric space} \]

**M-polyfold charts**

- near \( y_0: [-L,L] \to \mathbb{R}^n \)
  \[ \mathbb{R} \times H^1([-L,L], \mathbb{R}^n) \ni (\zeta, z) \mapsto \Phi(\zeta, z) \in \mathcal{E} \]
  \[ \Phi(\zeta, z) = (\zeta, y_0 + \zeta \cdot z) \]

- near \( (y^0, \eta^0) \)
  \[ \text{nbhd}(y^0, \eta^0) \subset |\mathcal{E}| \]
  \[ \text{retraction (splitting)} \]
  \[ \text{homeomorphism} \]
  \[ \Phi: (v, \xi, \eta, z) \mapsto (\mathbf{1} + v \beta) \left( \begin{array}{l} 0 \\ (0,0) \\ \eta \end{array} \right) (\xi + \eta) \]

**Obj \( \mathcal{E} \) = \bigcup_{i \in \mathcal{E}} U_i \cup \bigcup_{j \in \mathcal{E}} \Omega_j \]**

**transition maps** (\( \cong \) to \footnote{for \((s \times t): \text{Mor } \mathcal{E} \to \text{Obj } \mathcal{E} \)}

\[ U_j \supset \Psi_j^{-1}(\mathcal{O}_i(\mathcal{U}_i)) \to U_i \]

\[ (v, \xi, \eta, z, \zeta) \mapsto (\mathbf{1} + v \beta) \left( \begin{array}{l} 0 \\ (0,0) \\ \eta \end{array} \right) (\xi + \eta) \]

**Note:** \( \Psi_j^{\circ} \circ \Psi_j \) has same formula size \( \Psi_j \) projects along \( \mathbb{R} \cdot \beta \)

\[ \text{sc}^{-\infty} \]

\[ \text{sc}^{\infty} \]

\[ \text{sc}^{\infty} \]

\[ \text{sc}^{\infty} \]

\[ \text{sc}^{\infty} \]

\[ \text{sc}^{\infty} \]

**using fixed** \( R_j(v) = R_j(\psi) \)

\[ \text{sc}^{\infty} \text{ follows as for splitting when using exponential gluing profile } \]

\[ R_j(v) = e^{v \cdot e} \]
**Example:** $f: \mathbb{R}^n \to \mathbb{R}$ Morse function, let $f = 0$

$M_{\text{Morse}}(\mathbb{R}^n, \mathbb{R}^n) = \delta^{-1}(0)$ for $M$-polyfold Fredholm section $\delta$.

$|X| = \bigcup_{L > 0} H^1([-L, L], \mathbb{R}^n) \cup H^1((0, \infty), \mathbb{R}^n) \times H^1((-\infty, 0), \mathbb{R}^n)$

$\psi: (y, \gamma) \mapsto (y, \gamma f(y))$

$\delta: \gamma \mapsto \gamma f(\gamma) \mapsto \gamma f(\gamma)$

**$M$-polyfold bundle charts**

- **near $y_0: [-L, L] \to \mathbb{R}^n$**

  
  $\mathbb{R} \times H^1([-L, L], \mathbb{R}^n) = U_{\mathbb{R}} \times \Theta \times (\Theta, \Theta f, \Theta f(\gamma_0 + \gamma))$

  $\Theta f(\gamma_0 + \gamma)$

  $H^1([-L, L], \mathbb{R}^n)$

- **near $(y_0^0, y_0^1)$**

  $[0, v_0] \times H^1((0, \infty), \mathbb{R}^n) \times H^1((-\infty, 0), \mathbb{R}^n)$

  $(\pi_{\mathbb{R}} \times \pi_{\mathbb{R}}), v_0 \in \mathbb{R}$

  $\{v\} \times \text{im} \pi_{R(w)}$

  $\Theta(v_0, v_0)$

  $\Theta_{\text{rel}}(v_0, v_0) = 0$

  $\Theta_{\text{rel}}(3, 3) = 0$

  **defined for all $(3, 3) \in \mathbb{E}$**

  **"filled section"**

  $(\pi_{\mathbb{R}} \times \pi_{\mathbb{R}}), v_0 \in \mathbb{R}$

  $\pi_{\mathbb{R}} |_{\mathbb{R}} = \pi_{\mathbb{R}}$

  $[0, v_0] \times \mathbb{E} \times H^0((0, \infty), \mathbb{R}^n) \times H^0((-\infty, 0), \mathbb{R}^n)$

  $\mathbb{E}$

  $D_{\text{rel}}(3, 3) = (\Theta_{\text{rel}} f)(y_0^0 + \gamma_0, y_0^1 + \gamma_0, (y_0^0 + \gamma_0, y_0^1 + \gamma_0))$

  $D_{\text{rel}}(3, 3) = (\Theta_{\text{rel}} f)(y_0^0 + \gamma_0, y_0^1 + \gamma_0, (y_0^0 + \gamma_0, y_0^1 + \gamma_0))$

  $\Theta_{\text{rel}} (y_0^0 + \gamma_0, y_0^1 + \gamma_0) = 0$
Definition 6.1.4. An \textit{M-polyfold bundle} is an $\mathcal{C}^\infty$ surjection $p : \mathcal{Y} \to \mathcal{X}$ between two M-polyfolds together with a real vector space structure on each fiber $\mathcal{Y}_x := p^{-1}(x) \subset \mathcal{Y}$ over $x \in \mathcal{X}$ such that, for a sufficiently small neighbourhood $U \subset \mathcal{X}$ of any point in $\mathcal{X}$ there exists a \textbf{local sc-trivialization} $\Phi : \mathcal{Y} \supset p^{-1}(U) \to \mathcal{R}$. The latter is an $\mathcal{C}^\infty$ diffeomorphism to an \textit{sc-bundle retract} $\mathcal{R} = \bigcup_{p \in \mathcal{O}} \{p\} \times \mathcal{R}_p \subset \mathcal{E} \times \mathcal{F}$ that covers an M-polyfold chart $\phi : U \to \mathcal{O} \subset \mathcal{E}$ in the sense that $\text{pr}_\mathcal{O} \circ \Phi = \phi \circ p$, and preserves the linear structure in the sense that $\Phi|_{\mathcal{Y}_x} : \mathcal{Y}_x \to \{\phi(x)\} \times \mathcal{R}_\phi(x)$ is an isomorphism in every fiber over $x \in U$.

$$E_{(\nu, \lambda)} = \left\{ \begin{array}{l} H^0([\nu,\lambda], R^e) \\ H^0([0,\nu], R^e) \times H^0([\lambda,\infty], R^e) \end{array} \right\} \cong \text{im} \mathcal{P}_\nu \quad \mathcal{R} = \bigcup_{(\nu, \lambda) \in \mathcal{O}} \{ (\nu, \lambda) \} \times \text{im} \mathcal{P}_\nu$$

Definition 6.1.1. Let $\mathcal{O} \subset [0, \infty)^k \times \mathcal{E}$ be an sc-retract with corners in the sense of Definition 5.3.4, and let $\mathcal{F}$ be an sc-Banach space. Then a \textit{sc-bundle retract} over $\mathcal{O}$ in $\mathcal{F}$ is a family of subspaces $(\mathcal{R}_p \subset \mathcal{F})_{p \in \mathcal{O}}$ that are scale smoothly parametrized by $p \in \mathcal{O}$ in the following sense: There exists a \textit{sc-retraction of bundle type},

\begin{equation}
U \times \mathcal{F} \longrightarrow [0, \infty)^k \times \mathcal{E} \times \mathcal{F}, \quad (v, e, f) \mapsto (r(v, e), \Pi(v, e), f),
\end{equation}

given by a neat sc-retraction $r : U \to [0, \infty)^k \times \mathcal{E}$ with image $r(U) = \mathcal{O}$ and a family of linear projections $\Pi_{(v, e)} : \mathcal{F} \to \mathcal{F}$ that are parametrized by $(v, e) \in U$, and whose images for $p = (v, e) \in \mathcal{O}$ are the given subspaces $\Pi_p(\mathcal{F}) = \mathcal{R}_p$. 

\textbf{4}
Definition 6.2.8. An \( sc^\infty \) section \( s : \mathcal{X} \to \mathcal{Y} \) of an \( M \)-polyfold bundle is a \( sc\)-Fredholm section if \( s \) is regularizing in the sense of Definition 6.1.8 and for each \( x \in \mathcal{X}_x \) there is a local \( sc\)-trivialization \( \Phi : p^{-1}(U) \to \mathcal{R} \) in the sense of Definition 6.1.4 over a neighbourhood \( U \subset \mathcal{X} \) of \( x \) with \( \Phi(x, 0) = 0 \), such that \( \Phi \) has a Fredholm filling in the sense of Definition 6.2.7.

\[
[v, \xi] \mapsto (v_I, f(v, \xi))
\]

\[
\begin{array}{c}
[0, \nu_0) \times H^1(0, \nu_0) \to H^1(0, \nu_0) \\
(v, \xi, \nu) \mapsto \begin{cases}
\left[ (\mathbb{A}^* + \nu \mathbb{A}) (y^2 + \mathbb{A}) (\mathbb{A}^* + \nu \mathbb{A}) (y^2 + \mathbb{A}) \right] & \nu = 0 \\
\left[ \left( \mathbb{A}^* + \nu \mathbb{A} \right) \left( \mathbb{A}^* + \nu \mathbb{A} \right) \Theta_{\nu_0} (y^2 + \mathbb{A}) \right] & \nu = 0
\end{cases}
\end{array}
\]

Definition 6.2.7. Let \( s : \mathcal{O} \to \mathcal{R} \), \( s(p) = (p, f(p)) \) be an \( sc^\infty \) section of an \( M \)-polyfold bundle model \( \pi : \mathcal{R} \to \mathcal{O} \) as in Definition 6.1.1, whose base is an \( sc\)-retract \( \mathcal{O} \subset \{ 0, \infty \}^k \times \mathbb{E} \) containing \( 0 \in \{ 0, \infty \}^k \times \mathbb{E} \), and with fibers \( \mathcal{R}_p \subset \mathcal{F} \) for \( p \in \mathcal{O} \). Then a \( \mathcal{F} \) filling at \( 0 \) for \( s \) over \( \mathcal{O} \) consists of

- a \( sc \)-retraction of bundle type \( \mathcal{R} : \mathcal{U} \times \mathbb{E} \to \mathcal{U} \times \mathbb{E} \) such that \( \mathcal{U} \subset \{ 0, \infty \}^k \times \mathbb{E} \) is an open subset and \( \mathcal{R}(p, h) = (r(p), \Pi_p h) \) is an \( sc^\infty \) section of the bundle \( \mathcal{O} \times \mathcal{R} \) with \( \mathcal{R}(0, 0) = 0 \) and \( \mathcal{R}(0, 1) = \mathcal{O} \) for all \( p \in \mathcal{O} \),

- an \( sc^\infty \) map \( \tilde{f} : \mathcal{U} \to \mathbb{F} \) that is \( sc \)-Fredholm at \( 0 \) in the sense of Definition 6.2.4, with the following properties:

\[
\begin{array}{c}
\tilde{f}|_0 = f; \\
\text{if } p \in \mathcal{U} \text{ such that } \tilde{f}(p) \in \mathcal{R}(p), \text{ then } p = r(p), \text{ that is } p \in \mathcal{O}.
\end{array}
\]

(iii) The linearisation of the map \( [0, \infty)^k \times \mathbb{E} \to \mathbb{F} \), \( p \mapsto (\mathbb{I} - \Pi_p) \tilde{f}(p) \) at each \( p \in \mathcal{O} \)

\[
\ker D_p \mathcal{R} = Ker D_{\mathcal{R}(p)} \ni (v, \xi) \mapsto (v, \xi) - (v, \xi) \\
\ker \Pi_p = Ker \Pi_p
\]

\[
\mathbb{R}^k \times \mathbb{E} = T_p \mathcal{O} \oplus T_p \mathcal{O}_p / \mathbb{E}_p = T_p \mathcal{O}_p \\
\mathbb{E}_p = T_p \mathcal{O}_0 \oplus \mathbb{E}_p = T_p \mathcal{O}_0 \oplus \mathbb{E}_p = T_p \mathcal{O}_0 \\
\mathbb{I}_p \ni \mathbb{I}_p \\
\mathbb{I}_p \ni \mathbb{I}_p \\
\mathbb{I}_p \ni \mathbb{I}_p
\]

\[
D_{\mathcal{F}} = D(\mathbb{I}_p) - D((\mathbb{I} - \Pi_p) \mathcal{O}_0) = \Pi_p - D_{\mathcal{F}} + (\mathbb{I} - \Pi_p) \mathcal{O}_0
\]

\[
\begin{cases}
\left( \Pi_{\nu_0} \mathcal{O} \mathbb{I}_p \mathcal{O}_0 | \Pi_{\nu_0} \mathcal{O} \mathbb{I}_p \mathcal{O}_0 \right) : T_p \mathcal{O} \oplus T_p \mathcal{O}_0 \to \mathbb{E}_p \oplus \mathbb{E}_p \\
\mathbb{I}_p \ni \mathbb{I}_p
\end{cases}
\]

Note: \( \mathbb{I}_p \) isomorphism \( \Rightarrow \mathbb{D}_{\mathcal{F}} \) surjective \( \iff \Pi_{\nu_0} \mathcal{D}_{\mathcal{F}} | T_p \mathcal{O}_0 \) surjective
Definition 6.3.1. A scale smooth section \( s : \mathcal{X} \to \mathcal{Y} \) is called \textbf{transverse (to the zero section)} if for every \( x \in s^{-1}(0) \) the linearization \( D_x s : T_x \mathcal{X} \to T_x \mathcal{Y} \) is surjective. Here the linearization \( D_x s \) is represented by the differential \( D_\phi(x)(\Pi \circ f \circ r)|_{T_{\phi(x)} \mathcal{O}} : T_{\phi(x)} \mathcal{O} \to \Pi_{\phi(x)}(\mathbb{F}) \) in any local sc-trivialization \( p^{-1}(U) \supseteq \bigcup_{p \in \mathcal{O}} \Pi_p(\mathbb{F}) \) which covers \( \phi : \mathcal{X} \supset U \to \mathcal{O} = r(\mathcal{U}) \subset \mathbb{E} \) and transforms \( s \) to \( p \to (p, f(p)) \).

(by Note on p. 5)

\[ \mathcal{F} : \mathcal{U} \to \mathbb{F} \wedge \mathbb{O} \]

\[ \downarrow \text{I.F.Thm. of scale calculus} \]

Theorem 6.3.2 ([HWZ2], Thm. 5.14). Let \( s : \mathcal{X} \to \mathcal{Y} \) be a transverse sc-Fredholm section. Then the solution set \( \mathcal{M} := s^{-1}(0) \) inherits from its ambient space \( \mathcal{X} \) a smooth structure as finite dimensional manifold. Its dimension is given by the Fredholm index of \( s \) and the tangent bundle is given by the kernel of the linearized section, \( T_x \mathcal{M} = \ker D_x s \).

Theorem 6.3.7. ([HWZ2], Theorem 5.22) Let \( \pi : \mathcal{Y} \to \mathcal{X} \) be a strong M-polyfold bundle modeled on sc-Hilbert spaces, and let \( s : \mathcal{X} \to \mathcal{Y} \) be a proper Fredholm section.

(i) For any auxiliary norm \( N : \mathcal{Y}_1 \to [0, \infty) \) and neighbourhood \( s^{-1}(0) \subset \mathcal{U} \subset \mathcal{X} \) controlling compactness, there exists an \( sc^+ \)-section \( \nu : \mathcal{X} \to \mathcal{Y}_1 \) with \( \text{supp} \nu \subset \mathcal{U} \) and \( \sup_{x \in \mathcal{X}} N(\nu(x)) < 1 \), and such that \( s + \nu \) is transverse to the zero section. In particular, \( (s + \nu)^{-1}(0) \) carries the structure of a smooth compact manifold.

(ii) Given two transverse perturbations \( \nu_i : \mathcal{X} \to \mathcal{Y}_1 \) for \( i = 0, 1 \) as in (i), controlled by auxiliary norms and neighbourhoods \( (N, \mathcal{U}) \) controlling compactness, there exists an \( sc^+ \)-section \( \tilde{\nu} : \mathcal{X} \times [0, 1] \to \mathcal{Y}_1 \) such that \( \{(x, t) \in \mathcal{X} \times [0, 1] \mid s(x) + t \tilde{\nu}(x, t) \} \) is a smooth compact cobordism from \( (s + \nu_0)^{-1}(0) \) to \( (s + \nu_1)^{-1}(0) \).